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Dynamic analysis for the selection of parameters and initial population, in particle swarm optimization

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Abstract In this paper we consider the evolutionary Particle Swarm Optimization (PSO) algorithm, for the minimization of a *computationally costly* nonlinear function, in global optimization frameworks. We study a reformulation of the standard iteration of PSO (Clerc and Kennedy in IEEE Trans Evol Comput 6(1) 2002), (Kennedy and Eberhart in IEEE Service Center, Piscataway, IV: 1942–1948, 1995) into a linear dynamic system. We carry out our analysis on a generalized PSO iteration, which includes the standard one proposed in the literature. We analyze three issues for the resulting generalized PSO: first, for any particle we give both theoretical and numerical evidence on an efficient choice of the starting point. Then, we study the cases in which either deterministic and uniformly randomly distributed coefficients are considered in the scheme. Finally, some convergence analysis is also provided, along with some necessary conditions to avoid diverging trajectories. The results proved in the paper can be immediately applied to the standard PSO iteration.

Keywords Global optimization · Evolutionary optimization · Particle Swarm Optimization · Dynamic linear system · Convergence analysis

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1 Introduction

This paper is concerned with the solution of the optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{1.1}$$

where at present $f : \mathbb{R}^n \to \mathbb{R}$ is a *continuous* and *coercive function*. The latter hypothesis implies that the level sets of $f(x)$ are closed and bounded, so that (1.1) has solution. In particular, we aim at detecting a global minimum x^* of (1.1), such that $f(x^*) \le f(x)$, for any $x \in \mathbb{R}^n$, and the function is computationally costly.

The problem (1.1) is inherently difficult, since it requires an exhaustive exploitation of $f(x)$. This usually claims for a two phases approach: on one hand, a *global* search identifies suitable subsets where promising candidates of global minima are confined. Then, an efficient *local* search provides accurate approximations of each candidate, by exploring the corresponding subset. Thus, the resulting algorithm always includes the computational burden of both the *global* and the *local* phase.

Further difficulties arise when solving (1.1), in case a poor information on $f(x)$ is available; e.g., the derivatives of $f(x)$ are not available and the two phases above require a derivative free approach. The latter scenario becomes even dramatic whenever the computational burden for the function evaluation is relevant, too [24].

Examples of such difficult problems arise in several real applications, e.g., in the aerospace, automotive and naval engineering, where Navier-Stokes solvers for shape design are used. Here, the CPU time for one function evaluation may require up to several hours, depending on the geometry details [13,25].

As well known, unlike local optimization frameworks, there are not optimality conditions which can characterize the global minimum x^*. A large number of approaches have been proposed in the literature for the solution of (1.1) (see [24] and therein references), either adopting deterministic [12] and/or stochastic [11] techniques. Under suitable assumptions, deterministic methods exploit the local information on $f(x)$ and aim at extending their exploration over a wide and possibly *dense* subset of $\mathbb{R}^n$.

Similarly, stochastic techniques randomly generate a sequence $\{x_k\}$ approaching the solution, and usually provide asymptotic convergence results in terms of probability. A typical convergence result involves a subsequence of points $\{x_k\}_\mathcal{K}$ such that (see e.g., [24]) $Pr\left\{\lim_{k\to\infty,\, k\in\mathcal{K}} \rho(x_k, X^*) = 0\right\} = 1$, where X^* is the set of global minima of $f(x)$ in $\mathbb{R}^n$, $\rho(\cdot,\cdot)$ is a *distance* and $Pr\{\cdot\}$ indicates a measure of *probability*. Evidently, the use of the latter techniques requires the exploration over a possibly dense subset of $\mathbb{R}^n$; thus, the increasing computational burden would be definitely unaffordable for many optimization problems, e.g., shape design and structure optimization problems.

In the last decades both deterministic and stochastic methods for problem (1.1) were suggested by biological systems and/or social behavioural interpretations. In particular, the latter methods often disregard convergence analysis and focus on the heuristic solution of the problem in hand.

Examples of such iterative approaches are *genetic algorithms* and *evolutionary algorithms*, where a set of initial points (namely the *population* of *individuals/agents*) is chosen in the feasible set. Then, the population usually changes according with operations, which results from monitoring a suitable fitness function [19]. Genetic algorithms update the population with operations as *crossover (recombination)*, *selection* and *mutation* on each individual, inspired by biological paradigms [31] (see also the early reference [9]).

Evolutionary algorithms use the information collected by the entire population on the fitness function, in order to modify the population [28] (for more material and references see also the recent book [1]).

In this paper we analyze the PSO heuristics [15], an iterative method for global optimization, in the class of evolutionary algorithms. The latter scheme was originally conceived in 1995 as a model to describe the behavior of a flock of birds. Then, it has been recently improved (see also [5,6,22,29,36]) and is widely adopted to solve several applications [2,8,25,35].

The growing interest for PSO algorithm is encouraged by several features, which are definitely appealing in aerospace and naval engineering applications [13]: constant *computational cost* and *memory requirements* at each iteration, availability of a *reasonably approximate solution after few iterations*, the *derivatives* of the objective function are not required, possibly easy *parallelization*, etc. As remarked above, in ship design simulation the cost of each function evaluation may be huge, so that very few function evaluations are allowed. We observe that other state-of-art Evolutionary Algorithms (EA) show similar qualities (see for instance CMA-ES and the books [9,19]). Moreover, some numerical experiences (see for instance [30] and the website http://www.ntu.edu.sg/home/EPNSugan) on different test sets proved that PSO is comparable with them.

The general PSO iteration can be described at step $k \ge 0$ by the iterative scheme

$$x_j^{k+1} = x_j^k + v_j^k, \qquad j = 1, \dots, P, \tag{1.2}$$

where $x_j^k \in \mathbb{R}^n$ is the current position of the j-th individual (*particle*) of the population (*swarm*), $v_j^k \in \mathbb{R}^n$ is the search direction (*velocity*) and x_j^{k+1} is the new position at step $k+1$. Observe that v_j^k may be *not* a descent direction for the objective function $f(x)$ at x_j^k. In particular, the direction v_j^k depends on both the search direction v_j^{k-1} and the paths of the particles. The PSO version in [15] considers at step k the direction

$$v_j^k = v_j^{k-1} + \alpha^k \otimes \left(p_j^k - x_j^k\right) + \beta^k \otimes \left(p_g^k - x_j^k\right), \tag{1.3}$$

where α^k, β^k are suitable random n-vectors, '$\otimes$' means component-by-component product and

$$p_j^k = \arg\min_{0\le h\le k} \left\{ f(x_j^h) \right\}, \qquad j = 1, \dots, P$$
$$p_g^k = \arg\min_{0\le h\le k,\ j=1,\dots,P} \left\{ f(x_j^h) \right\}.$$

The velocity v_j^{k-1} in (1.3) contributes to maintain also at step k, the velocity of step $k-1$. With α^k (the *cognitive parameter*) the contribution from the history of the j-th particle, summarized in the vector $p_j^k - x_j^k$, is suitably considered. On the other hand, β^k (the *social parameter*) weights the influence of all the particles in the swarm (the vector $p_g^k - x_j^k$) on the j-th particle.

In this paper we figure out a partial analysis of the particles trajectories, on a very general reformulation of PSO iteration. Our approach is not completely new in the literature (similar issues are partially considered and investigated in [6,15,20,22,26,32–34]). First, we extend several results from the latter papers and [3]; then, we provide both theoretical and numerical evidence in order to assess the starting point of each particle (again see [3]). Finally, we study the use of either deterministic or stochastic parameters in PSO, within a preliminary framework of convergence.

In this paper **C** represents the set of complex numbers. With ‘rk(A)’ we indicate the rank of matrix A, with ‘tr(A)’ we indicate its trace, ‘det(A)’ is its determinant and I the identity matrix of suitable dimension. Finally, given the vectors $a, b \in \mathbb{R}^n$, we denote by $a \otimes b$ the vector whose i-th entry is given by $a_i b_i$.

In Sect. 2 we describe our generalized PSO iteration by means of a dynamic linear system, whose properties are partially analyzed in Sects. 3, 4 and 5. Section 6 is devoted to investigate promising starting points for the particles. Section 7 considers some theoretical and numerical issues about the use of deterministic or stochastic parameters, and Sect. 8 provides some partial convergence results. In particular, the Sect. 8 addresses the PSO parameters selection by imposing that the particles trajectories are confined in a suitable compact set. A section of Conclusions and an “Appendix” complete the paper.

2 A dynamic model for PSO

Consider formulae (1.2)-(1.3), let $\alpha^k = c_j^k r_j^k$ and $\beta^k = c_g^k r_g^k$, where c_j^k, c_g^k are real coefficients and r_j^k, r_g^k are vectors in $[0, 1]^n$. Then, introducing in (1.3) also the real parameters χ and w^k, we obtain the following iteration of PSO:

$$\begin{cases} v_j^{k+1} = \chi \left[w^k v_j^k + c_j^k r_j^k \otimes \left(p_j^k - x_j^k \right) + c_g^k r_g^k \otimes \left(p_g^k - x_j^k \right) \right], & k = 0, 1, \dots, \\ x_j^{k+1} = x_j^k + v_j^{k+1}, & k = 0, 1, \dots, \end{cases} \tag{2.1}$$

where $j = 1, \dots, P$ indicates the j-th particle, P is finite, v_j^k and x_j^k are the *velocity* and the *position* of particle j at step k, p_j^k and p_g^k, respectively satisfy

$$p_j^k = \arg\min_{l \le k} \left\{ f(x_j^l) \right\}, \qquad p_g^k = \arg\min_{l \le k,\ j=1,\dots P} \left\{ f(x_j^l) \right\}, \tag{2.2}$$

and $\chi, w^k, c_j^k, r_j^k, c_g^k, r_g^k$ are bounded coefficients. *We assume at present that the latter parameters are real constants* with $0 \le r_j^k \le 1$ and $0 \le r_g^k \le 1$. The latter position will be discussed in Sect. 7, *where we analyze the introduction of uniformly distributed random parameters*.

Then, we can generalize (2.1) by assuming that possibly the velocity v_j^{k+1} depends on all the terms $p_h^k - x_j^k$, $h = 1, \dots, P$. This corresponds to allow a more general *social* contribution in the PSO iteration. The new iteration (see also [18]) is therefore

$$\begin{cases} v_j^{k+1} = \chi_j^k \left[w_j^k v_j^k + \sum_{h=1}^{P} c_{h,j}^k r_{h,j}^k \otimes \left(p_h^k - x_j^k \right) \right], & k = 0, 1, \dots, \\ x_j^{k+1} = x_j^k + v_j^{k+1}, & k = 0, 1, \dots, \end{cases} \tag{2.3}$$

where $c_{h,j}^k$ and $r_{h,j}^k$ depend on the step (k), the current particle (j) and the other particles (h).

The latter generalization is not so common in the literature. Anyway, (2.3) includes (2.1) and we give theoretical results in Sect. 4 for the generalized formulation (2.3).

Without loss of generality at present we focus on the j-th particle and omit the subscript in the recurrence (2.3). Moreover, we assume in (2.3) $\chi_j^k = \chi, c_{h,j}^k = c_h, r_{h,j}^k = r_h$ and $w_j^k = w$, for any $k \ge 0$. This hypothesis will be removed *in Sect. 4 and Sect. 7*, where *we will extend our results including the case of non-constant and random parameters*, for any k.

With the latter position the iteration (2.3) is equivalent to the *dynamic, linear* and *stationary system* (se also [3])

$$X(k+1) = \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix} X(k) + \begin{pmatrix} \sum_{h=1}^{P} \chi c_h r_h p_h^k \\ \sum_{h=1}^{P} \chi c_h r_h p_h^k \end{pmatrix}, \tag{2.4}$$

where

$$X(k) = \begin{pmatrix} v^k \\ x^k \end{pmatrix} \in \mathbb{R}^{2n}, \qquad k \geq 0.$$

The sequence $\{X(k)\}$ identifies a trajectory in the state space $\mathbb{R}^{2n}$, and since (2.4) is a linear and stationary system, we may consider the *free response* $X_L(k)$ and the *forced response* $X_F(k)$ of the trajectory $\{X(k)\}$. Then, considering (2.4) we explicitly obtain at step $k \geq 0$ [27]

$$X(k) = X_L(k) + X_F(k), \tag{2.5}$$

where

$$X_L(k) = \Phi(k) X(0), \qquad X_F(k) = \sum_{\tau=0}^{k-1} H(k-\tau) U(\tau), \tag{2.6}$$

and (after few calculation)

$$\Phi(k) = \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix}^k \in \mathbb{R}^{2n \times 2n}, \tag{2.7}$$

$$H(k-\tau) = \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix}^{k-\tau-1} \in \mathbb{R}^{2n \times 2n}, \tag{2.8}$$

$$U(\tau) = \begin{pmatrix} \sum_{h=1}^{P} \chi c_h r_h p_h^\tau \\ \sum_{h=1}^{P} \chi c_h r_h p_h^\tau \end{pmatrix} \in \mathbb{R}^{2n}. \tag{2.9}$$

Remark 2.1 Observe that $X_L(k)$ in (2.6) does not depend on the vector p_h^k, but uniquely on the initial point $X(0)$. Conversely, $X_F(k)$ in (2.6) depends on the vector p_h^k and is independent of $X(0)$.

The latter remark allows us to compute separately the two terms $X_L(k)$ and $X_F(k)$. In order to carry out our conclusions, in the next two sections we focus on $X_L(k)$ by investigating the eigenpairs of matrix $\Phi(k)$ in (2.7).

3 Computation of the free response $X_L(k)$

The computation of the trajectory $\{X(k)\}$ in (2.5) strongly depends on the sequences $\{p_h^k\}$, $h = 1, \dots, P$. Furthermore, if the eigenvalues of $\Phi(k)$ in (2.6) are suitably chosen, relation

$$\lim_{k\to\infty} X_L(k) = 0 \tag{3.1}$$

holds, i.e., the free response $X_L(k)$ is effective only for finite values of k. Nevertheless, in the Introduction we reported some classes of shape design problems where the latter feature is relevant, since the computational resources only allow modest values of k. In this section we determine the conditions which yield (3.1) (see also [3] and [33]).

For the sake of simplicity let us consider the following position in (2.7) and (2.8)

$$a = \chi w, \qquad \omega = \sum_{h=1}^{P} \chi c_h r_h, \tag{3.2}$$

so that for any $\lambda \in \mathbf{C}$

$$\Phi(1) - \lambda I = \begin{pmatrix} (a-\lambda)I & -\omega I \\ aI & (1-\omega-\lambda)I \end{pmatrix},$$

whose Schur complement is given by $(1-\omega-\lambda+a\omega/(a-\lambda))I$. We aim at computing the $2n$ eigenvalues $\lambda^{(1)}, \dots, \lambda^{(2n)}$ of $\Phi(1)$; thus, assuming provisionally $\lambda^{(l)} \neq a, l = 1, \dots, 2n$, we obtain

$$\begin{aligned}\det(\Phi(1) - \lambda I) &= \det \begin{pmatrix} I & 0 \\ \dfrac{a}{a-\lambda} I & I \end{pmatrix} \det \begin{pmatrix} (a-\lambda)I & -\omega I \\ 0 & \left(1-\omega-\lambda+\dfrac{a\omega}{a-\lambda}\right) I \end{pmatrix} \\ &= (a-\lambda)^n \left[(1-\omega-\lambda)+\frac{a\omega}{a-\lambda}\right]^n = \left[\lambda^2-(1-\omega+a)\lambda+a\right]^n . \end{aligned} \tag{3.3}$$

Therefore, $\Phi(1)$ has at most two distinct eigenvalues, each with algebraic multiplicity n, i.e.,

$$\begin{aligned} \lambda^{(1)} = \cdots = \lambda^{(n)} = \lambda_1 &= \frac{1-\omega+a-\left[(1-\omega+a)^2-4a\right]^{1/2}}{2} \\ \lambda^{(n+1)} = \cdots = \lambda^{(2n)} = \lambda_2 &= \frac{1-\omega+a+\left[(1-\omega+a)^2-4a\right]^{1/2}}{2}. \end{aligned} \tag{3.4}$$

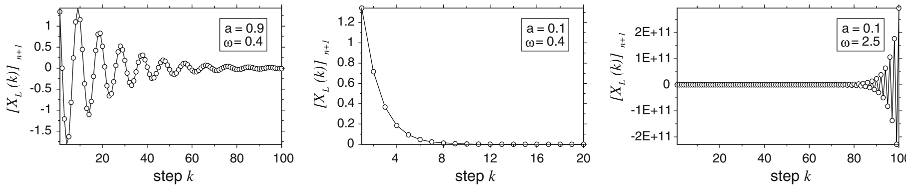

Fig. 1 The $(n+1)$-th entry of $X_L(k)$, with $0 < a < 1$ and, respectively, $\left(1-\sqrt{a}\right)^2 < \omega < \left(1+\sqrt{a}\right)^2$ *-left-*, $0 < \omega < \left(1-\sqrt{a}\right)^2$ *-center-* and $\omega > \left(1+\sqrt{a}\right)^2$ *-right-*

Observe that either $\lambda_1 = a$ or $\lambda_2 = a$ *if and only if* $a = 0$ or $\omega = 0$. However, $a \neq 0$ and $\omega \neq 0$ is a very common assumption in PSO literature [6,15]. Thus, without loss of generality $\lambda_1 \neq a$ and $\lambda_2 \neq a$, and in accordance with the literature we may consider the following.

Assumption 3.1 We assume in (3.2) $a \neq 0$ and $\omega > 0$.

Moreover, $p_\Phi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ is the minimal polynomial of matrix $\Phi(1)$ and $\det[\Phi(1)] = a^n$, so that for $n = 1$ we obtain $\det[\Phi(1)] = \lambda_1\lambda_2 = a$. By considering the quantity $\Delta = (1 - \omega + a)^2 - 4a$ in (3.4), the following cases are analyzed:

- $a < 0$ which yields $\Delta > 0$, thus λ_1, λ_2 are *real* and *distinct* (one is positive and the other is negative);
- $a > 0$ which yields condition $\Delta \geq 0$ as long as (see Assumption 3.1)

$$0 < \omega \leq \left(1-\sqrt{a}\right)^2 \qquad \text{or} \qquad \omega \geq \left(1+\sqrt{a}\right)^2, \tag{3.5}$$

which generates the following subcases:

(1) $0 < \omega < \left(1-\sqrt{a}\right)^2 \implies \lambda_1$ and λ_2 are real, distinct and both positive;

(2) $\omega = \left(1-\sqrt{a}\right)^2$ or $\omega = \left(1+\sqrt{a}\right)^2 \implies \lambda_1 = \lambda_2 = \dfrac{1-\omega+a}{2} = \pm\sqrt{a}$;

(3) $\omega > \left(1+\sqrt{a}\right)^2 \implies \lambda_1$ and λ_2 are real, distinct and both negative.

In terms of dynamic analysis [27] we have the situation in Fig. 1, where

- if $a > 0$ and $\left(1-\sqrt{a}\right)^2 < \omega < \left(1+\sqrt{a}\right)^2$ then λ_1 and λ_2 are complex conjugate (*left*), so that they generate natural pseudoperiodic modes (this generalizes the results in [14]). If $a > 0$ and $0 < \omega < \left(1-\sqrt{a}\right)^2$ then we have exactly $2n$ natural aperiodic modes (*center*). Finally, if $a > 0$ and $\omega > \left(1+\sqrt{a}\right)^2$ then we have exactly $2n$ natural alternating modes (*right*);
- if $a < 0$ we have exactly n natural aperiodic modes and n natural alternating modes.

According with (2.6) (see [27]),

$$X_L(k) = \Phi(k)X(0) = \Phi(1)^k X(0) = \begin{pmatrix} aI & -\omega I \\ aI & (1-\omega)\,I \end{pmatrix}^k X(0), \tag{3.6}$$

and from Assumption 3.1 $\lambda_q \neq a$, $q = 1, 2$, then

$$\text{rk}\left[\Phi(1) - \lambda_q I\right] = \text{rk}\left[\begin{pmatrix} I & 0 \\ \dfrac{a}{a-\lambda_q} I & I \end{pmatrix}\begin{pmatrix} (a-\lambda_q)I & -\omega I \\ 0 & 0 \end{pmatrix}\right] = n,$$

which implies that the *algebraic multiplicity* and the *geometric multiplicity* of eigenvalues λ_1 and λ_2 coincide with n, i.e., the unsymmetric matrix $\Phi(1)$ is *diagonalizable* and a basis of $2n$ real eigenvectors of $\Phi(1)$ exists. The latter result will be used in Sect. 5 to determine the initial population in PSO.

4 Conditions for non-diverging trajectories

In this section we consider the iteration (2.3), where the parameters $\chi_j^k, w_j^k, c_{h,j}^k, r_{h,j}^k$, are possibly *non-constant* at each iteration[1] $k \geq 0$. Thus, here we address a more general case with respect to the previous sections. Let the parameter w_j^k in (2.3) be given by

$$w_j^k = \phi_j(k) w_j^a + \left[1 - \phi_j(k)\right] w_j^b, \tag{4.1}$$

where $0 < \phi_j(k) \leq 1$, $w_j^b \geq 0$, $w_j^a \geq 0$. The expression in (4.1) includes the proposals for the coefficient w_j^k in [6,15,29,36], where there is not full agreement about both the role and the way to assess w_j^k.

Regardless of possible heuristic interpretations of the role played by parameter w_j^k in (2.3), here we analyze the computation of the eigenvalues λ_1 and λ_2 in (3.4), when we assume that the coefficient w_j^k is possibly calculated as in (4.1). Therefore, for the j-th particle the coefficient a in (3.2) (as well as the coefficient ω and the eigenvalues λ_1 and λ_2) is affected by w_j^k, so that at step k we have to consider in (3.2) the parameters

$$a_j^k = \chi_j^k w_j^k, \qquad \omega_j^k = \sum_{h=1}^{p} \chi_j^k c_{h,j}^k r_{h,j}^k \tag{4.2}$$

in place of a and ω. In addition, from (2.3) to (2.9) *a necessary condition in order to impose non-diverging trajectories* is given by the following

Lemma 4.1 *Consider the eigenvalues λ_1 and λ_2 in (3.4) and let*

$$\begin{cases} |\lambda_1| < 1 \\ |\lambda_2| < 1, \end{cases} \tag{4.3}$$

then

$$\lim_{k \to \infty} X_L(k) = 0, \tag{4.4}$$

regardless of the choice of the initial point $X(0)$.

Thus, for the j-th particle, from (3.3) $\lambda_1 \lambda_2 = a_j^k$, and from Lemma 4.1 we have to impose at each iteration conditions (4.3), in order to obtain (4.4). Therefore the parameter a_j^k should satisfy

$$\left| a_j^k \right| < 1. \tag{4.5}$$

[1] This implies that the linear system (2.3) may not be time-invariant any more for the j-th particle. Anyway, from the *causality* property of system (2.6) [27], for any particle we can equivalently consider at step k a new linear system, with the same parameters of (2.6), and whose starting point is $X(k-1)$ instead of $X(0)$. For the particle j the vector $X(k-1)$ is computed at step $k-1$ of (2.6), with the parameters $\chi_j^{k-1}, w_j^{k-1}, c_{h,j}^{k-1}, r_{h,j}^{k-1}$.

Proposition 4.2 *Consider iteration* (2.3) *and relation* (4.2)*, let at step* $k \geq 0$

$$\begin{aligned} 0 < \left|a_j^k\right| < 1, \qquad & j = 1, \ldots, P \\ 0 < \omega_j^k < 2(a_j^k + 1), \qquad & j = 1, \ldots, P, \end{aligned} \tag{4.6}$$

with $\omega_j^k \neq \left(1 \pm \sqrt{a_j^k}\right)^2$ *for any* $a_j^k > 0$*. Then, for any* $k \geq 0$ *and* $1 \leq j \leq P$ *we have* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *in relation (4.3).*

Proof We distinguish two cases: $-1 < a_j^k < 0$ and $0 < a_j^k < 1$. In the first case, imposing $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and considering that λ_1 and λ_2 are real, after some calculation we obtain for $k \geq 0$ and $1 \leq j \leq P$ conditions

$$-1 < a_j^k < 0, \qquad 0 < \omega_j^k < 2(a_j^k + 1).$$

In the second case we consider two sub-cases:

(i) $0 < a_j^k < 1, \;\; 0 < \omega_j^k < \left(1 - \sqrt{a_j^k}\right)^2$ or $\left(1 + \sqrt{a_j^k}\right)^2 < \omega_j^k < 2(a_j^k + 1)$,

(ii) $0 < a_j^k < 1, \;\; \left(1 - \sqrt{a_j^k}\right)^2 < \omega_j^k < \left(1 + \sqrt{a_j^k}\right)^2$.

If (i) holds λ_1 and λ_2 are real, so that again a long but trivial computation yields $|\lambda_1| < 1$ and $|\lambda_2| < 1$. On the other hand, if (ii) holds then from (3.4) λ_1 and λ_2 are complex conjugate with $|\lambda_1| = |\lambda_2| = \sqrt{a_j^k} < 1$. This completes the proof. □

Remark 4.1 From Proposition 4.2 observe that (4.5) is not a sufficient condition to impose non-diverging particle trajectories. Moreover, the particle trajectory might be non-diverging *as long as* an index $\bar{k}$ exists, such that the hypotheses in Proposition 4.2 hold for $k \geq \bar{k}$. Clearly for $\bar{k} = 0$ we again obtain the results of Proposition 4.2. On this guideline the results of this section extend those in [6,29,32].

5 The starting point of particles

In this section for the sake of simplicity we consider again (2.4)–(2.9), where the PSO coefficients are independent of the iteration index k. According with the last considerations of Sect. 3, $2n$ *real* eigenvectors of matrix $\Phi(1)$ exist and without loss of generality they have the form

$$v_i = \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix}, \qquad v_i^1, v_i^2 \in \mathbb{R}^n, \quad i = 1, \ldots, 2n.$$

Then, to compute the eigenvectors of $\Phi(1)$ we impose relations

$$0 = \left[\Phi(1) - \lambda^{(i)} I\right] v_i = \begin{pmatrix} I & 0 \\ \dfrac{a}{a - \lambda^{(i)}} I & I \end{pmatrix} \begin{pmatrix} (a - \lambda^{(i)}) I & -\omega I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix},$$

so that from Assumption 3.1 we can explicitly obtain the $2n$ eigenvectors. It can be readily seen that these eigenvectors are linearly independent, provided that $\lambda_1 \neq \lambda_2$; the latter condition is not restrictive for our purposes, therefore from (3.5) we consider the following.

Assumption 5.1 We assume in (3.2) $\omega \neq \left(1-\sqrt{a}\right)^2$ and $\omega \neq \left(1+\sqrt{a}\right)^2$ for any $a > 0$, so that the $2n$ eigenvectors of matrix $\Phi(1)$ are linearly independent.

Therefore, after few calculation the matrix $V = \left[v_1 \ldots v_n v_{n+1} \ldots v_{2n}\right] \in \mathbb{R}^{2n\times 2n}$ exists, such that $V^{-1}\Phi(1)V = \Lambda$, with

$$\Lambda = \begin{pmatrix} \lambda_1 I & \\ & \lambda_2 I \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$V = \begin{pmatrix} I & I \\ \frac{a-\lambda_1}{\omega} I & \frac{a-\lambda_2}{\omega} I \end{pmatrix} \quad \text{and} \quad V^{-1} = \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} (a-\lambda_2)I & -\omega I \\ -(a-\lambda_1)I & \omega I \end{pmatrix}.$$

The previous results yield from (3.6)

$$\begin{aligned} X_L(k) &= V\Lambda^k V^{-1} X(0) \\ &= \frac{1}{\lambda_1-\lambda_2}\sum_{i=1}^{n}\Big[\lambda_1^k X(0)^T\left((a-\lambda_2)e_i - \omega e_{n+i}\right)v_i \\ &\quad -\lambda_2^k X(0)^T\left((a-\lambda_1)e_i - \omega e_{n+i}\right)v_{n+i}\Big] \\ &= \frac{1}{\lambda_1-\lambda_2}\sum_{i=1}^{n}\Big[\lambda_1^k\left((a-\lambda_2)X(0)_i - \omega X(0)_{n+i}\right)v_i \\ &\quad -\lambda_2^k\left((a-\lambda_1)X(0)_i - \omega X(0)_{n+i}\right)v_{n+i}\Big], \end{aligned}$$

where $e_i \in \mathbb{R}^{2n}$ is the unit vector with 1 in position i, and $X(0)_i$ is the i-th entry of $X(0)$, i.e.,

$$X(0) = \begin{pmatrix} X(0)_1 \\ \vdots \\ X(0)_{2n} \end{pmatrix}.$$

Furthermore, for $i = 1, \ldots, n$

$$\begin{cases} v_i = e_i + \dfrac{a-\lambda_1}{\omega} e_{n+i} \\ v_{n+i} = e_i + \dfrac{a-\lambda_2}{\omega} e_{n+i}, \end{cases}$$

then

$$X_L(k) = \sum_{i=1}^{n}\left[\gamma_1(k)X(0)_i e_i - \gamma_2(k)X(0)_{n+i}e_i + \gamma_3(k)X(0)_i e_{n+i} - \gamma_4(k)X(0)_{n+i}e_{n+i}\right], \tag{5.1}$$

where the coefficients $\gamma_i(k), i = 1, \ldots, 4$ are given in Table 1. Observe that the first value of each coefficient $\gamma_1(k), \ldots, \gamma_4(k)$ refers to real eigenvalues λ_1 and λ_2, while the second value holds in case λ_1 and λ_2 are conjugate (i.e., $\lambda_1 = \rho e^{-j\theta}$, $\lambda_2 = \rho e^{j\theta}$, with $a = \rho^2$).

Table 1 The coefficients $\gamma_i(k)$, $i = 1, \ldots, 4$ in (5.1)

$$\gamma_1(k) = \begin{cases} \dfrac{\lambda_1^k(a-\lambda_2) - \lambda_2^k(a-\lambda_1)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \text{ real} \\ \rho^{k+1}\dfrac{\sin k\theta}{\sin\theta} - \rho^k \dfrac{\sin(k-1)\theta}{\sin\theta} & \lambda_1, \lambda_2 \text{ complex conjugate,} \end{cases}$$

$$\gamma_2(k) = \begin{cases} \dfrac{\omega(\lambda_1^k - \lambda_2^k)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \text{ real} \\ \omega\rho^{k-1}\dfrac{\sin k\theta}{\sin\theta} & \lambda_1, \lambda_2 \text{ complex conjugate,} \end{cases}$$

$$\gamma_3(k) = \begin{cases} \dfrac{(\lambda_1^k - \lambda_2^k)}{\lambda_1 - \lambda_2}\dfrac{(a-\lambda_1)(a-\lambda_2)}{\omega} & \lambda_1, \lambda_2 \text{ real} \\ \rho^k \dfrac{\sin k\theta}{\sin\theta}\left(\dfrac{\rho^2 - 2\rho\cos\theta + 1}{\omega}\right) & \lambda_1, \lambda_2 \text{ complex conjugate,} \end{cases}$$

$$\gamma_4(k) = \begin{cases} \dfrac{\lambda_1^k(a-\lambda_1) - \lambda_2^k(a-\lambda_2)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \text{ real} \\ \rho^{k+1}\dfrac{\sin k\theta}{\sin\theta} - \rho^k \dfrac{\sin(k+1)\theta}{\sin\theta} & \lambda_1, \lambda_2 \text{ complex conjugate.} \end{cases}$$

In the end, from (5.1) we obtain

$$X_L(k) = \sum_{i=1}^{n} \left[\gamma_1(k)X(0)_i - \gamma_2(k)X(0)_{n+i}\right] e_i + \left[\gamma_3(k)X(0)_i - \gamma_4(k)X(0)_{n+i}\right] e_{n+i}. \tag{5.2}$$

Remark 5.1 Relation (5.2) suggests the following remarkable conclusions:

- the free response $X_L(k)$ in (2.5) has the same formal expression for each particle;
- by simply imposing $X(0)_i = X(0)_{n+i} = 0$ in (5.2), the free response of a particle has zero entries on the i-th and $(n+i)$-th axis. Thus, each particle's trajectory has nonzero projection on any subspace of $\mathbb{R}^{2n}$, provided that the initial point $X(0)$ is suitably chosen.
- if the eigenvalues λ_1 and λ_2 are complex conjugate, i.e., $\lambda_1 = \rho e^{-j\theta}$ and $\lambda_2 = \rho e^{j\theta}$, then the phase θ is given by

$$\theta = \text{arctg}\left[\left(\frac{4a}{(1-\omega+a)^2} - 1\right)^{1/2}\right]. \tag{5.3}$$

  Thus, the circular trajectories have the frequency $\theta/(2\pi)$, which might be suitably set in order to widely explore the state space.

The importance of the latter considerations relies on the role played by $X_L(k)$ and $X_F(k)$ in the overall trajectory. From the expression of the coefficients $\gamma_1(k), \ldots, \gamma_4(k)$ in relation (5.2), it is clearly confirmed that if $|\lambda_1| \geq 1$ or $|\lambda_2| \geq 1$ we might have $\|X_L(k)\|_2 \to \infty$ for any infinite subsequence of k.

From Lemma 4.1, observe that the free response $X_L(k)$ of each particle does not affect asymptotically the sequence $\{X(k)\}$. However, a suitable choice of the starting point $X(0)$ of any particle may guarantee an improved exploration of the state space. The latter issue is noteworthy and plays a key role within global optimization frameworks [24].

In the light of Lemma 4.1 the next proposition provides, under mild assumptions, a relation between the index k and the quantities $\|X_L(k)\|_1$, $\|X_L(0)\|_1$. We prove the latter result for a wide set of parameters a and ω in (3.2), which is relevant in the applications.

Proposition 5.1 *Let* $0 < a < 1$ *and* $\left(1-\sqrt{a}\right)^2 < \omega < \left(1+\sqrt{a}\right)^2$ *in (3.2). Let* $0 < \epsilon_k < 1$ *with* $k \geq 0$*, and assume*

$$k \geq 1 + \frac{\ln\left(\frac{\omega|\sin\theta|\epsilon_k}{8}\right)}{\ln\rho} \qquad \text{for any } 0 < \omega < 1,$$
$$k \geq 1 + \frac{\ln\left(\frac{|\sin\theta|\epsilon_k}{8}\right)}{\ln\rho} \qquad \text{for any } \omega \geq 1, \tag{5.4}$$

where $\lambda_1 = \rho e^{-j\theta}$ *and* $\lambda_2 = \rho e^{j\theta}$ *are the eigenvalues of* $\Phi(1)^k$*. Then,*

$$\|X_L(k)\|_1 \leq \epsilon_k \|X_L(0)\|_1 . \tag{5.5}$$

Proof From (3.3) the hypothesis straightforwardly yields $\rho = \sqrt{a} < 1$ and $0 < \omega < 4$. Then, from (5.1)

$$\|X_L(k)\|_1 \leq \sum_{i=1}^{n} \left\{ \frac{\rho^{k-1}}{|\sin\theta|} \left[2\,|X_L(0)_i| + 4\,|X_L(0)_{n+i}|\right] + \frac{\rho^k}{|\sin\theta|}\left[\frac{4}{\omega}\,|X_L(0)_i| + 2\,|X_L(0)_{n+i}|\right]\right\},$$

which yields

$$\|X_L(k)\|_1 \leq \frac{4\rho^{k-1}}{\omega|\sin\theta|}\left[2\,\|X_L(0)\|_1\right] = \frac{8}{\omega}\frac{\rho^{k-1}}{|\sin\theta|}\|X_L(0)\|_1, \quad \text{for } 0 < \omega < 1,$$
$$\|X_L(k)\|_1 \leq \frac{4\rho^{k-1}}{|\sin\theta|}\left[2\,\|X_L(0)\|_1\right] = 8\frac{\rho^{k-1}}{|\sin\theta|}\|X_L(0)\|_1, \quad \text{for } \omega \geq 1.$$

Then, after few calculation we see that the inequalities

$$\frac{8}{\omega}\frac{\rho^{k-1}}{|\sin\theta|}\|X_L(0)\|_1 \leq \epsilon_k\|X_L(0)\|_1, \quad \text{for } 0 < \omega < 1,$$
$$8\frac{\rho^{k-1}}{|\sin\theta|}\|X_L(0)\|_1 \leq \epsilon_k\|X_L(0)\|_1, \quad \text{for } \omega \geq 1,$$

can be respectively satisfied if relations (5.4) hold. □

Finally observe that the equivalence among norms, provides a relation between the quantities $\|X_L(k)\|_2$ and $\|X_L(0)\|_2$, similar to (5.5).

6 Hints for the starting point of each particle

According with Remark 5.1, in this section we give both some theoretical and numerical indications about the choice of the initial point $X(0)$ of each particle: we will also give evidence that it is a crucial issue in PSO (see also [21,23] and therein references). Let us indicate $X(k)^{(j)}$ the trajectory of the j-th particle. From (2.5) $X(k)^{(j)}$ linearly depends on the contribution of the free response $X_L(k)^{(j)}$, i.e., it depends on the initial point $X(0)^{(j)}$.

At present we assume that exactly n particles compose the swarm, though our results will be extended later on to smaller/larger sets of particles. Given n particles in the swarm, we aim

at assessing the initial points $X(0)^{(j)}$, $j = 1, \dots, n$, in such a way that the state space $\mathbb{R}^{2n}$ is explored as widely as possible by the trajectories $\{X(k)^{(j)}\}$. This is a relevant and intriguing issue in general for global optimization algorithms, which often resort to randomly generated or general purpose sequences of initial points (see also [17] and therein references). Here we propose a set of starting points $\{X(0)^{(j)}\}$ such that for any fixed index k, the sequence $\{X_L(k)^{(j)}\}$ is *scattered* in the state space $\mathbb{R}^{2n}$.
To the latter purpose we consider in (3.2), for any particle $j = 1, \dots, n$, the parameters a_j, w_j in place of a, w. Accordingly, the coefficients $\gamma_1(k), \dots, \gamma_4(k)$ in (5.2) are given for the j-th particle by $\gamma_1(k)^{(j)}, \dots, \gamma_4(k)^{(j)}$.

Consider the scalars $\alpha^{(j)}, \beta^{(j)}, \sigma^{(j)} \in \mathbb{R}$, such that $|\alpha^{(j)}| + |\beta^{(j)}| \neq 0$, $\sigma^{(j)} > 0$, and the vector $t_j \in \mathbb{R}^n$, $j = 1, \dots, n$, with

$$t_j = \sigma^{(j)} \left[\frac{\sqrt{n}}{n} \sum_{i=1}^{n} e_i - \frac{\sqrt{n}}{2} e_j \right]. \tag{6.1}$$

Then, let the vector $X(0)^{(j)}$ be given by

$$X(0)^{(j)} = \begin{pmatrix} \alpha^{(j)} t_j \\ \beta^{(j)} t_j \end{pmatrix}, \qquad j = 1, \dots, n. \tag{6.2}$$

Proposition 6.1 *Consider the set of initial points $\{X(0)^{(j)}\}$ defined in (6.2). Consider in (3.2) for any particle j the parameters a_j, w_j in place of parameters a, w, and let the Assumption 5.1 hold. Then for any $k \geq 0$ the following relations hold*

$$\left[X_L(k)^{(j)} \right]^T X_L(k)^{(h)} = 0, \qquad \text{for any } \; 1 \leq j \neq h \leq n. \tag{6.3}$$

Proof From Assumption 5.1 relation (5.2) is defined. Thus, by simple substitution from (6.1) and (6.2), we obtain

$$X_L(k)^{(j)} = \begin{pmatrix} \left[\alpha^{(j)} \gamma_1(k)^{(j)} - \beta^{(j)} \gamma_2(k)^{(j)}\right] t_j \\ \left[\alpha^{(j)} \gamma_3(k)^{(j)} - \beta^{(j)} \gamma_4(k)^{(j)}\right] t_j \end{pmatrix}$$

$$X_L(k)^{(h)} = \begin{pmatrix} \left[\alpha^{(h)} \gamma_1(k)^{(h)} - \beta^{(h)} \gamma_2(k)^{(h)}\right] t_h \\ \left[\alpha^{(h)} \gamma_3(k)^{(h)} - \beta^{(h)} \gamma_4(k)^{(h)}\right] t_h \end{pmatrix},$$

and since $t_j^T t_h = 0$, for $1 \leq j \neq h \leq n$, relations (6.3) hold. □

Observe that the choice (6.1, 6.2) generates at step k the n vectors $X_L(k)^{(1)}, \dots, X_L(k)^{(n)}$, which form an orthogonal basis in the state subspace of the positions. In Figs. 2, 3 we show examples with $n = 2$ (i.e., $X_L(k)^{(j)} \in \mathbb{R}^4$, $j = 1, 2$) and two particles (Particles #1 and #2 are sketched in the two dimensional subspace of positions with empty triangles and filled circles, respectively; we set the values $a_1 = a_2 = 0.9$ and $\omega_1 = \omega_2 = 0.4$ in (3.2) for both particles). On the left side of the figures only the free responses $X_L(k)^{(1)}$ and $X_L(k)^{(2)}$ are drawn, while on the right side the full trajectories $X(k)^{(1)}$ and $X(k)^{(2)}$ are reported. Moreover, in Figs. 2*a* and 3*a* the initial points $X(0)^{(1)}$ and $X(0)^{(2)}$ are randomly chosen, while in Figs. 2*b* and 3*b* $X(0)^{(1)}$ and $X(0)^{(2)}$ are chosen as in (6.2) (this explains the perfect orthogonality of the free responses). Observe that in the first cases the algorithm fails to converge, because the trajectories substantially move along the segment joining $X(0)^{(1)}$ and $X(0)^{(2)}$. On the contrary, in Figs. 2*b* and 3*b* the orthogonality of the free responses of the two particles determines the convergence (Particle #2 is attracted by Particle #1 but from (6.3) they do not overlap). The latter result may be substantially interpreted as follows. *At least for small values of parameter k, the choice (6.2) tends to preserve orthogonality among the trajectories*. This helps the particles to be 'distributed' in the state space, inasmuch as for any k the set of positions $\left\{x_1^k, \ldots, x_P^k\right\}$ (see (2.3)) is likely an independent set. Note that the latter analysis holds both in case $x^* \neq 0$ (Fig. 2) and $x^* = 0$ (Fig. 3).

Finally, we can readily prove that replacing the sequence $\{t_j\}$ in (6.1) with *any orthogonal sequence* of n nonzero vectors, the result of Proposition 6.1 still holds.

From a geometrical viewpoint, the j-th initial point distributed as in (6.1, 6.2) belongs to a *spherical surface* of radius $\sqrt{n}\sigma^{(j)}/2$ (we set in the computation $\sigma^{(j)} = j \cdot \sigma/n$ with σ random in [0, 1]). This is consistent with the idea of both "scattering" the particles at the first iteration, and preserving a promising pattern in the following ones. On the contrary, the random initialization tends to distribute the initial points in a *volume*, consistently with the idea of only "scattering" the particles at the first iteration.

In the "Appendix" we provide a numerical experience where PSO is applied for the solution of 61 test problems, of size n ($2 \leq n \leq 30$). The first 51 problems (from the literature) are listed below (the suffix '_mod' stands for 'modified', meaning that the vector of unknowns x is shifted as in $x \leftarrow (x - \pi/4)$)

Func. num.	Name	n	Func. num.	Name	n
1	Six humps camel back	2	2	Treccani	2
3	Quartic	2	4	Schubert	2
5	Schubert pen.1	2	6	Schubert pen.2	2
7	Shekel5	4	8	Shekel7	4
9	Shekel10	4	10	Exponential	2
11	Exponential	4	12	Cosine mixture	2
13	Cosine mixture	4	14	Hartman3	3
15	Hartman6	6	16	Levy 5^n loc. min.	2
17	Levy 5^n loc. min.	5	18	Levy 5^n loc. min.	10
19	Levy 5^n loc. min.	20	20	Levy 10^n loc. min.	2
21	Levy 10^n loc. min.	5	22	Levy 10^n loc. min.	10
23	Levy 10^n loc. min.	20	24	Levy 15^n loc. min.	2
25	Levy 15^n loc. min.	5	26	Levy15^n loc. min.	10
27	Levy 15^n loc. min.	20	28	Griewank	2
29	Griewank	5	30	Griewank	10
31	Griewank	20	32	Levy 5^n loc. min.	30
33	Levy 10^n loc. min.	30	34	Levy 15^n loc. min.	30
35	Griewank	30	36	Odessa6	2
37	Odessa11	2	38	Odessa18	2
39	Levy3	2	40	Goldstein-Price	2
41	Freudenstein-Roth	2	42	Odessa99	2
43	Griewank_mod	2	44	Griewank_mod	5
45	Griewank_mod	10	46	Griewank_mod	20
47	Treccani_mod	2	48	Cosine mixture_mod	2
49	Cosine mixture_mod	4	50	Exponential_mod	2
51	Exponential_mod	4			

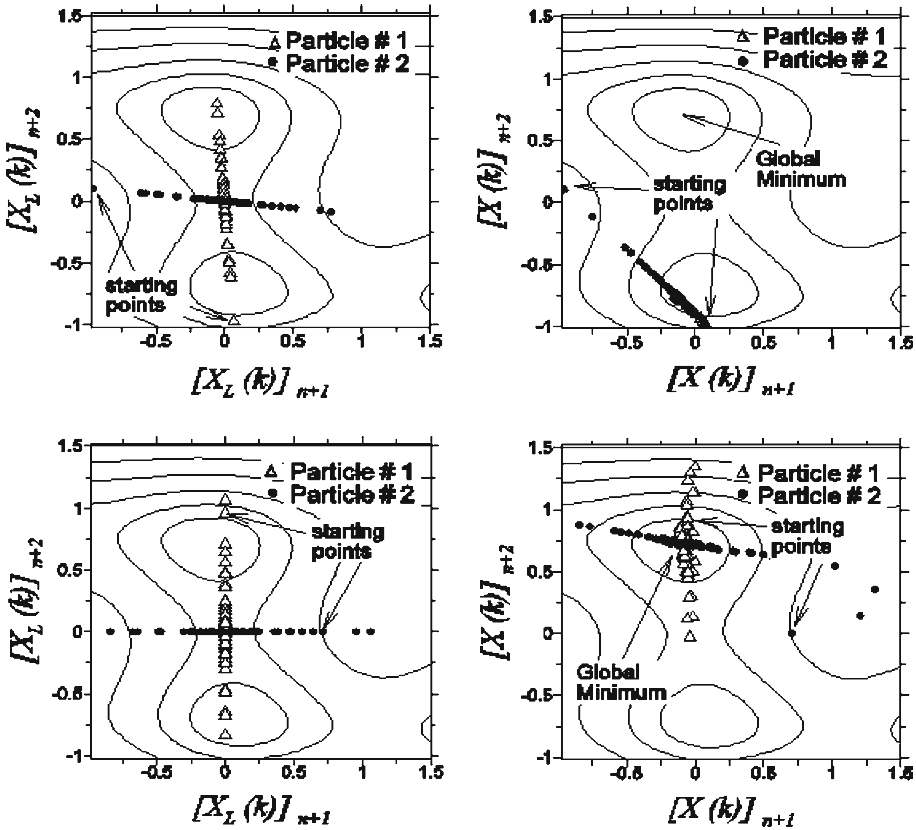

Fig. 2 (*Above* **a**) Random choice of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$ (*'Six-humps camel-back'* function [7]). Particle #1 and Particle #2 substantially overlap: the global minimum is not detected. (*Below* **b**) Choice (6.2) of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$. The two particles generate independent trajectories: global minimum is detected. Here $x^* \neq 0$.

while the last 10 test problems (from [30]) are given by

Func. num.	Name	n
52	Shifted Schwefels Problem 1.2	10
53	Shifted Rotated High Conditioned Elliptic Function	10
54	Shifted Schwefels Problem 1.2 with Noise in Fitness	10
55	Schwefels Problem 2.6 with Global Optimum on Bounds	10
56	Shifted Rosenbrocks Function	10
57	Shifted Rotated Griewanks Function without Bounds	10
58	Shifted Rotated Ackleys Function with Global Optimum on Bounds	10
59	Shifted Rastrigins Function	10
60	Shifted Rotated Rastrigins Function	10
61	Shifted Rotated Weierstrass Function	10

We compare two scenarios applying PSO: the random initialization of the particles and our proposal (6.1, 6.2). Note that our main focus is on studying and testing (6.1, 6.2) for the initial position of the particles. Thus, instead of assessing questionable parameters and using the latter in our numerical experience, we simply adopted the parameters proposed in [32],

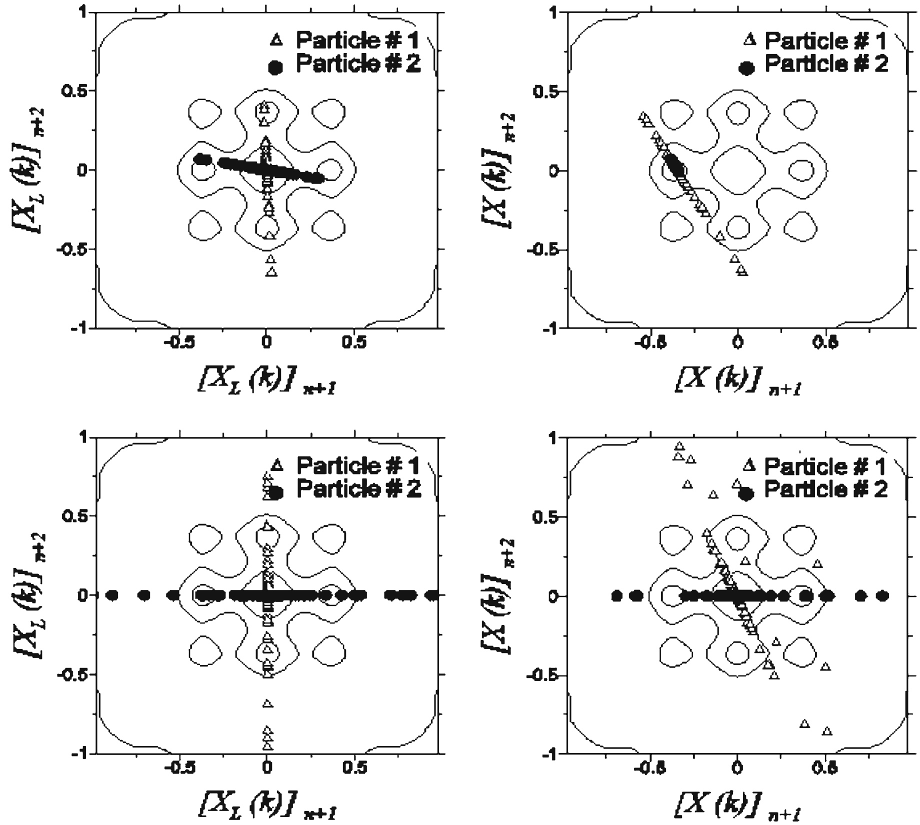

Fig. 3 (*Above* **a**) Random choice of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$ (*'Cosine mixture'* function [7]). Particle #1 and Particle #2 substantially overlap: the global minimum is not detected. (*Below* **b**) Choice (6.2) of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$. As in Fig. 2 the two particles generate independent trajectories: global minimum is detected. Here $x^* = 0$

setting $\chi = 1$, $w^k = 0.729$, $c_j^k = 1.494$, $c_g^k = 1.494$. Furthermore, since our approach is tailored for problems where the objective function is computationally costly, we allowed up to respectively, $20n + 1$, $40n + 1$ and $80n + 1$ function evaluations. The quantities reported in the tables of section Appendix are

- x_{rand} identifies the scenario with random initialization;
- x_{orth} identifies the scenario with our initialization in (6.1, 6.2);
- f^* is the value of $f(x)$ in the global minimum;
- *nf_av* is the average number of function evaluations over 25 runs;
- *f_bst* is given by $\min_{1 \le i \le 25} \{f_i^*\}$, where f_i^* is the best value of $f(x)$ found in the i-th run;
- *f_av* is given by $\sum_{i=1}^{25} f_i^*/25$, where f_i^* is the best value of $f(x)$ found in the i-th run;
- *f_wst* is given by $\max_{1 \le i \le 25} \{f_i^*\}$, where f_i^* is the best value of $f(x)$ found in the i-th run;
- *st. dev.* is the standard deviation given by $\left(\sum_{i=1}^{25} \left(f_i^* - f^*\right)^2 /25\right)^{0.5}$, where f_i^* is the best value of $f(x)$ found in the i-th run.

In the tables the best results are in bold, and the asterisks indicate a failure when the result exceeds the value 10^6 (though no particles diverged in the numerical experience).

When *a few* (i.e., $20n + 1$) function evaluations are allowed (equivalently, as long as the free response $X_L(k)$ of the dynamic linear system (2.4) is significantly bounded away from zero), our proposal is preferable over the 61 problems. The same still holds even when $40n+1$ and $80n+1$ function evaluations are allowed. However, in the latter cases, considering the last 10 test functions our proposal seems comparable with the random initialization. Moreover, in problems 54 and 57 the maximum number of function evaluations is outreached when the current solution is far from the real one, so that we have some failures in both the scenarios. Over 8 out of the last 10 problems, since the algorithms stop when $\|X_L(k)\|/\|X_L(0)\|$ is still large (see Proposition 5.1), we deduce that a larger number of function evaluations should be allowed, in order to get more precise solutions. A partial extension of our results, which considers more than n particles in the swarm, was described in [4]. Finally, we observe that our approach can be possibly combined with the schemes proposed in [23] and [21].

6.1 The extended case

This section considers a possible extension of the results reported in Sect. 6. Let $Q \in \mathbb{R}^{n\times n}$ and suppose Q is orthogonal, i.e., $Q^T Q = I_n$. In particular, the columns $\{q_1, \dots, q_n\}$ of Q represent an orthonormal basis of $\mathbb{R}^n$. Suppose we *rotate* each column-vector q_k of Q by a specific angle, into the vector s_k, $k = 1, \dots, n$. Then, an orthogonal matrix $\bar{Q} \in \mathbb{R}^{n\times n}$ exists such that the vectors $s_k = \bar{Q} q_k$, $1 \le k \le n$, are still an orthonormal basis.

We introduce a measure d of the distance between the two bases $\{q_k\}$ and $\{s_k\}$, according with

$$d = \|Q - \bar{Q}Q\|_F, \tag{6.4}$$

where $\|\cdot\|_F$ indicates the *Frobenius norm* of a matrix, i.e., $\|A\|_F = [\mathrm{tr}(A^T A)]^{1/2}$. Then, we can consider the orthogonal matrix Q^* which satisfies

$$Q^* = \arg\max_{\bar{Q}} \|Q - \bar{Q}Q\|_F, \qquad \text{s.t.} \;\; \bar{Q}^T \bar{Q} = I_n. \tag{6.5}$$

In other words, according with the definition of the distance d in (6.4), we say that the orthogonal matrix Q^*Q is at *maximum distance* from the matrix Q. We recall that by definition we have for any $A, B \in \mathbb{R}^{n\times n}$

$$\|A - B\|_F^2 = \mathrm{tr}\left[(A - B)^T (A - B)\right] = \mathrm{tr}\left(A^T A\right) + \mathrm{tr}\left(B^T B\right) - 2\mathrm{tr}\left(A^T B\right) \tag{6.6}$$

thus, $Q^* = -I_n$ satisfies (6.5) since

$$\mathrm{tr}\left(Q^T Q\right) = \mathrm{tr}\left[\left(Q^* Q\right)^T \left(Q^* Q\right)\right] = n,$$

so that

$$\left\| Q - Q^* Q \right\|_F^2 = 2n - 2\mathrm{tr}\left(Q^T Q^* Q\right) = 4n. \tag{6.7}$$

From (6.4, 6.5), the latter result yields the intuitive consideration that the bases $\{q_1, \dots, q_n\}$ and $\{-q_1, \dots, -q_n\}$ are maximally distant in the sense of formula (6.4). Therefore, since the columns of Q are orthogonal vectors, we could use the sets $\{q_1, \dots, q_n\}$ and $\{-q_1, \dots, -q_n\}$ in place of the vectors in (6.1). To sum up, if only $2n$ particles are chosen for the swarm, then from (6.1) and considering the Proposition 6.1 we suggest the following set of initial particles' position ($1 \le j \le n$)

$$X(0)^{(j)} = \begin{pmatrix} 0 \\ t_j \end{pmatrix}, \qquad t_j = 2(-1)^j \sigma^{(j)} \left[\frac{\sqrt{n}}{n} \sum_{i=1}^{n} e_i - \frac{\sqrt{n}}{2} e_j \right],$$
$$X(0)^{(n+j)} = \begin{pmatrix} 0 \\ t_{n+j} \end{pmatrix}, \quad t_{n+j} = 2(-1)^{j+1} \sigma^{(n+j)} \left[\frac{\sqrt{n}}{n} \sum_{i=1}^{n} e_i - \frac{\sqrt{n}}{2} e_j \right]. \tag{6.8}$$

We recall that by a short calculation the matrices $(t_1 \ldots t_n)$ and $(t_{n+1} \ldots t_{2n})$ are orthogonal.

7 Use of random coefficients in PSO algorithm

In (2.3) we considered a very general PSO iteration, which includes as special cases the proposals in [6,15]. Furthermore, in iteration (2.3) we provisionally assumed that the coefficients $r_{h,j}^k$, $h, j = 1, \ldots, P$ were *real constants* in [0, 1], for any $k \ge 0$. In this section we remove the latter assumption and consider that for $h, j = 1, \ldots, P$, the coefficient $r_{h,j}^k$ is a *random parameter* with *uniform distribution* in the interval [0, 1]. We aim at recasting the results of the previous sections with the latter new position[2] (see also [26] on this issue).

On this guideline observe that in particular the analysis in Sects. 3-4 still holds, since the inequalities involved are also satisfied with $r_{h,j}^k$ randomly distributed in [0, 1]. Indeed, whenever (4.6) holds with (see (4.2)) $r_{h,j}^k = 1$, then it also holds if $r_{h,j}^k$ is randomly distributed in [0, 1], since a_j^k does not depend on $r_{h,j}^k$.

We only urge to precise, reasoning as in the proof of Proposition 4.2, that replacing '$r_{h,j}^k = 1$' with '$r_{h,j}^k$ randomly distributed in [0, 1]', may modify the natural modes associated with eigenvalues λ_1 and λ_2 in (3.4). Thus, Lemma 4.1 and Proposition 4.2 still hold: this result details more accurately the conclusions in [14], i.e., *the above introduction of randomness for coefficients* $r_{h,j}^k$, $h = 1, \ldots, P$, *seems hardly responsible for instability on its own*. However, at step k random coefficients may cause complex eigenvalues λ_1 and λ_2 to become real, so that the particles trajectory may be modified during the iterations. This also implies that if the free response of the trajectory (2.3) of particle j is non-diverging with $r_{h,j}^k = 1$ (i.e., $\lim_{k\to\infty} \left\| X(k)^{(j)} \right\| < +\infty$), then from Proposition 4.2 it hardly diverges with $r_{h,j}^k$ uniformly distributed in [0, 1]. (See for instance the trajectories in Fig. 4 with $n = 2$, where $r_{h,j}^k = 1$ -empty circles- and $r_{h,j}^k$ is randomly chosen in [0, 1] -filled circles-. The left side reports $[X_{L2}(k)]_4$, i.e., the fourth component of the *free response*, for the second particle; the right side reports $[X_2(k)]_4$, i.e., the overall fourth component of the second particle). The latter conclusion is an immediate consequence of relation (4.6), where the sequences $\left\{\omega_j^k\right\}$ (and not directly $\left\{r_{h,j}^k\right\}$) are responsible to yield relation (4.3). We remark that in Fig. 4, for a correct comparison, we used the same PSO parameters for all the test functions. Furthermore, observe that according with the literature, the free response of the *Six-humps camel-back* function is reported in the range $[-2, +2]$.

On the contrary (see Fig. 5), in some cases where the improper choice of deterministic coefficients in (2.3) causes diverging trajectories (empty circles), the use of randomly chosen coefficient $r_{h,j}^k$ in [0, 1] *may* yield convergence (filled circles). Observe that as expected the

[2] The contents of footnote 1 apply here too, with trivial adaptations.

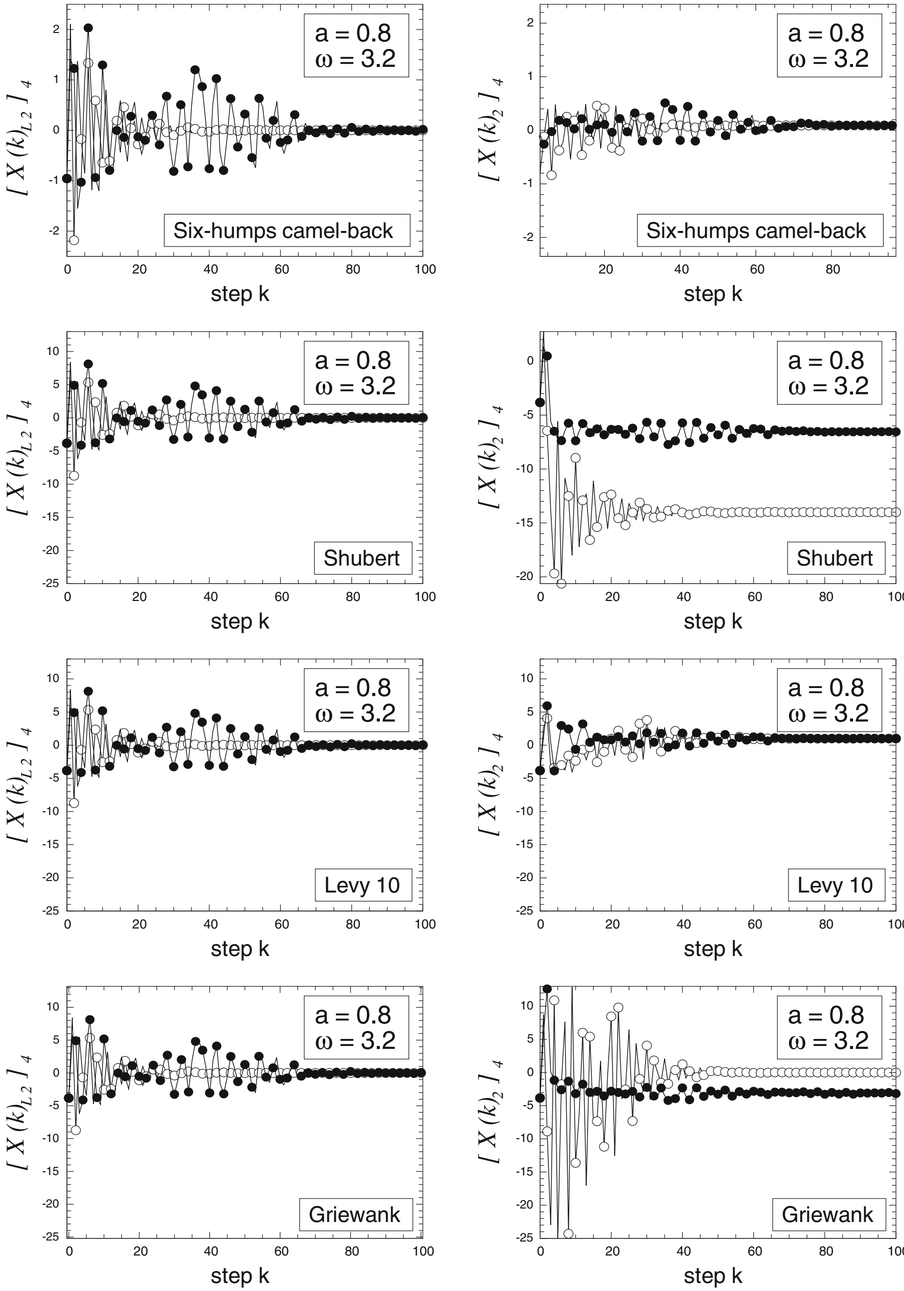

Fig. 4 Comparison of the converging path of the component $\left[x_2^k\right]_2$ (i.e., the second component of the second particle), when choosing in (2.3) $r_{h,j}^k = 1$ (*empty circles*) and $r_{h,j}^k$ randomly distributed in [0, 1] (*filled circles*). Parameters a and ω in (3.2) are reported for four test functions. On the *left side* we have the free response $[X_{L2}(k)]_4$, which does not depend on the function; on the *right side* $[X_2(k)]_4$ is reported

free responses in Figs. 4 and 5 are the same, regardless of the test function, according with (2.6)-(2.7).

Of course these conclusions do not include any statistical analysis on the trajectory of the particles. In particular, we guess that a detailed investigation of the statistical properties of the sequences $\left\{v_j^k\right\}$ and $\left\{x_j^k\right\}$, $j = 1, \ldots, P$, in terms of the random sequences $\left\{r_{h,j}^k\right\}$, could provide helpful indications on determining the probability of convergence to global minima [24]. In our opinion the latter issue deserves an appropriate study (see also [26]).

8 Preliminary convergence results

In this section we focus on the last topic of this paper. We consider the problem (1.1) and we study, under suitable assumptions, some preliminary properties of convergence for the sequences $\left\{x_j^k\right\}$, $j = 1, \ldots, P$, in (2.3). Observe that if the level set

$$\mathcal{L}_0 = \left\{x \in \mathbb{R}^n \ : \ f(x) \le f(x_0)\right\}, \qquad x_0 = arg\ max_{1\le j\le P} \left\{f(x_j^0)\right\},$$

is closed and bounded, then the sequences $\left\{p_j^k\right\}$ trivially satisfy the conditions

(1) $\left\{p_j^k\right\} \subseteq \mathcal{L}_0$,

(2) $\left\{p_j^k\right\}$ admit limit points for any $j = 1, \ldots, P$.

However, here we want to study the general case when no assumptions hold on $f(x)$, except the continuity. Thus, no assumption is considered for the level set $\mathcal{L}_0$. Here we also give conditions on the sequences $\left\{\chi_j^k\right\}$, $\left\{w_j^k\right\}$, $\left\{c_{h,j}^k\right\}$, $\left\{r_{h,j}^k\right\}$ in (2.3), in order to prevent from diverging trajectories.

We also highlight that in accordance with the Sect.2, here we substantially impose in our analysis some *conservative* conditions on the forced response $X_F(k)$. In particular, in order to solve (1.1) with PSO, we have to guarantee that suitable PSO parameters exist such that the properties (1–2) above hold.

Assumption 8.1 Consider the global optimization problem (1.1) and the iteration (2.3) with the positions (2.2). Let $\hat{k} \ge 0$ be an index such that for $j = 1, \ldots, P$, $\left\|x_j^{\hat{k}}\right\| \le d$, $\left\|v_j^{\hat{k}}\right\| \le d$ and $\left\|p_j^{\hat{k}}\right\| \le d$, with $d < +\infty$. Let $0 \le \epsilon < 1$ and

$$\hat{\mathcal{L}} = \left\{y \in \mathbb{R}^n \ : \ \|y\| \le \frac{d}{1-\epsilon}\right\}, \tag{8.1}$$

with $f(x)$ *continuous* on $\hat{\mathcal{L}}$. Assume that for any $k \ge \hat{k}$ and $j = 1, \ldots, P$ the sequences $\left\{\chi_j^k\right\}$, $\left\{w_j^k\right\}$, $\left\{c_{h,j}^k\right\}$, $\left\{r_{h,j}^k\right\}$ satisfy:

(H1) $\left|\chi_j^k w_j^k\right| \le \epsilon < 1$,

(H2) $$\sum_{h=1}^{P} \left|\chi_j^k c_{h,j}^k r_{h,j}^k\right| \le \frac{\left(\epsilon - \left|\chi_j^k w_j^k\right|\right)\epsilon^{k-\hat{k}}}{2/(1-\epsilon)}, \qquad h = 1, \ldots, P, \ p_h^k \ne x_j^k.$$

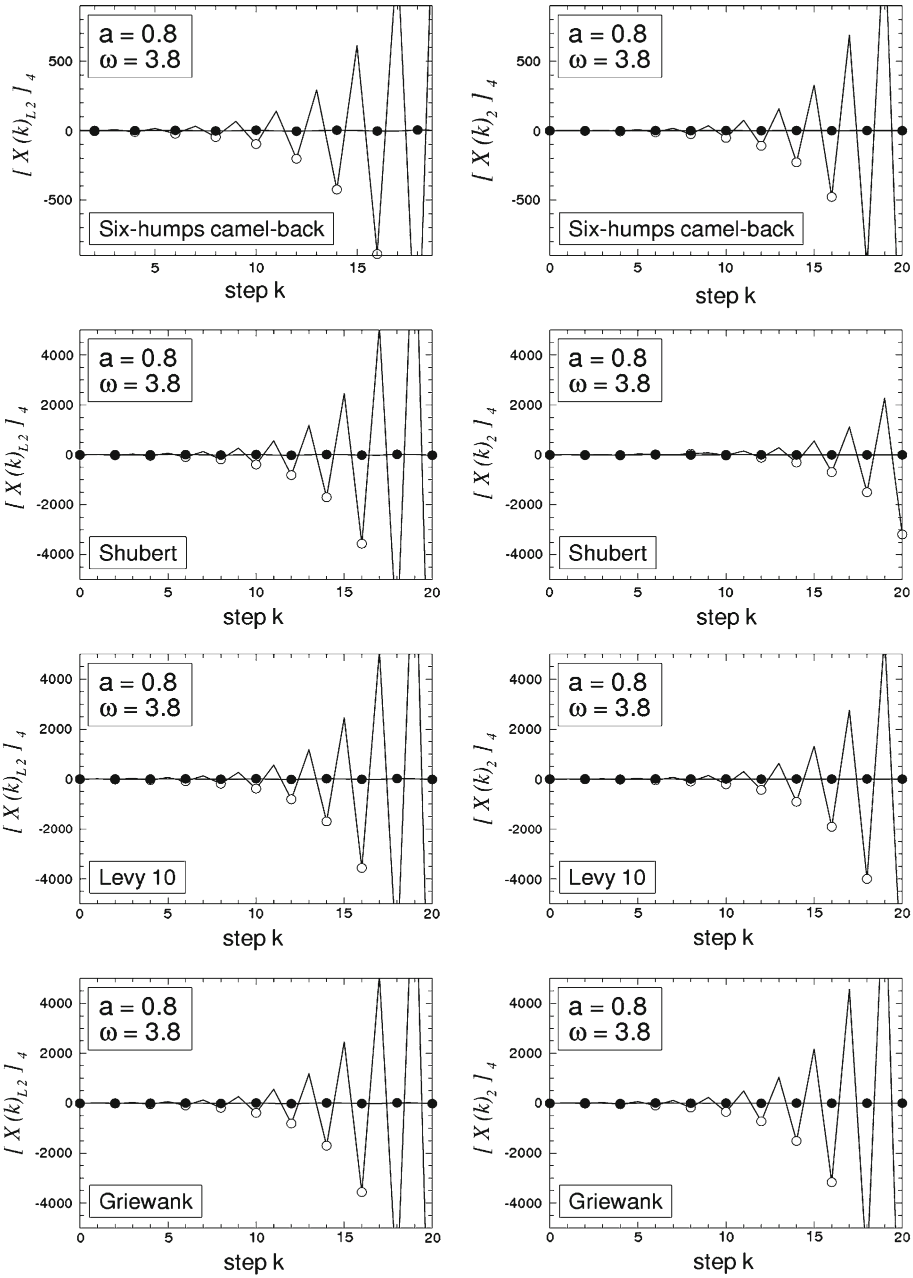

Fig. 5 Path of the component $\left[x_2^k\right]_2$ (i.e., the second component of the second particle) when choosing in (2.3) $r_{h,j}^k = 1$ (*diverging empty circles*) and $r_{h,j}^k$ randomly distributed in [0, 1] (*converging filled circles*). Parameters a and ω in (3.2) are reported for four test functions. On the *left side* we have the free response $[X_{L2}(k)]_4$, which does not depend on the function; on the *right side* $[X_2(k)]_4$ is reported

First note that the set $\hat{\mathcal{L}}$ is closed and bounded. The next theorem proves that when solving (1.1), the PSO iteration (2.3) generates for $k \geq \hat{k}$ sequences of points $\left\{x_j^k\right\}$ such that $\left\{x_j^k\right\} \subseteq \hat{\mathcal{L}}$.

Theorem 8.1 *Consider iteration (2.3) and suppose Assumption 8.1 holds. Then, for $k \geq \hat{k}$ we have:*

(a) $\left\{x_j^k\right\} \subset \hat{\mathcal{L}}, \ \left\{p_j^k\right\} \subset \hat{\mathcal{L}}, \qquad j = 1, \ldots, P,$

(b) $\left\|v_j^{k+1}\right\| \leq \epsilon^{k+1-\hat{k}} d, \qquad j = 1, \ldots, P,$

(c) $\lim_{k\to\infty} \left\|v_j^{k+1}\right\| = 0, \qquad j = 1, \ldots, P.$

Proof Observe that by definition $p_h^k \in \mathcal{P}_h^k = \left\{z \in \mathbb{R}^n \, : \, z = \arg\min_{\hat{k}\leq l\leq k} \left\{f\left(x_h^l\right), f\left(p_h^{\hat{k}}\right)\right\}\right\}$, $h = 1, \ldots, P$ and by the definition of v_j^{k+1} in (2.3) we have

$$\left\|v_j^{k+1}\right\| \leq \left|\chi_j^k w_j^k\right| \left\|v_j^k\right\| + \left\|\sum_{h=1}^{P} \chi_j^k c_{h,j}^k r_{h,j}^k \left(p_h^k - x_j^k\right)\right\|, \quad k \geq \hat{k}, \ j = 1, \ldots, P. \tag{8.2}$$

Now we prove *(a)* and *(b)* by induction. For $k = \hat{k}$ and $j = 1, \ldots, P$ we have

$$\left\|v_j^{\hat{k}+1}\right\| \leq \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right| d + \sum_{h=1}^{P} \left|\chi_j^{\hat{k}} c_{h,j}^{\hat{k}} r_{h,j}^{\hat{k}}\right| \left\|p_h^{\hat{k}} - x_j^{\hat{k}}\right\| \leq \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right| d + \frac{\left(\epsilon - \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right|\right) 2d}{2/(1-\epsilon)} \leq \epsilon d,$$

$$\left\|x_j^{\hat{k}+1}\right\| \leq \left\|x_j^{\hat{k}}\right\| + \left\|v_j^{\hat{k}+1}\right\| \leq (1+\epsilon) d,$$

and $\left\|p_j^{\hat{k}+1}\right\| \leq \max\left\{\left\|p_j^{\hat{k}}\right\|, \left\|x_j^{\hat{k}+1}\right\|\right\} = (1+\epsilon)d$; thus, *(a)* and *(b)* hold with $k = \hat{k}$. Suppose now $x_j^k, p_j^k \in \hat{\mathcal{L}}$ and $\left\|v_j^k\right\| \leq \epsilon^{k-\hat{k}} d$, for $k > \hat{k}$; then, from Assumption 8.1 and relation (8.2):

$$\begin{aligned}
\left\|v_j^{k+1}\right\| &\leq \left|\chi_j^k w_j^k\right| \left\|v_j^k\right\| + \frac{\left(\epsilon - \left|\chi_j^k w_j^k\right|\right) \epsilon^{k-\hat{k}} \left(\frac{2}{1-\epsilon}\right) d}{\left(\frac{2}{1-\epsilon}\right)} \\
&\leq \left|\chi_j^k w_j^k\right| \epsilon^{k-\hat{k}} d + \left(\epsilon - \left|\chi_j^k w_j^k\right|\right) \epsilon^{k-\hat{k}} d = \epsilon^{k+1-\hat{k}} d,
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\left\|x_j^{k+1}\right\| &= \left\|x_j^{\hat{k}} + \sum_{h=\hat{k}+1}^{k+1} v_j^h\right\| \leq \left\|x_j^{\hat{k}}\right\| + \sum_{h=\hat{k}+1}^{k+1} \left\|v_j^h\right\| \leq d + \sum_{h=\hat{k}+1}^{k+1} \epsilon^{h-\hat{k}} d \\
&\leq d + \frac{d}{\epsilon^{\hat{k}}} \left(\sum_{h=0}^{k+1} \epsilon^h - \sum_{h=0}^{\hat{k}} \epsilon^h\right) = d + \frac{d}{\epsilon^{\hat{k}}} \left(\frac{1-\epsilon^{k+2}}{1-\epsilon} - \frac{1-\epsilon^{\hat{k}+1}}{1-\epsilon}\right) \\
&= d + \frac{d}{\epsilon^{\hat{k}}} \left(\frac{\epsilon^{\hat{k}+1} - \epsilon^{k+2}}{1-\epsilon}\right) = d + \frac{\epsilon - \epsilon^{k+2-\hat{k}}}{1-\epsilon} d \leq d + \frac{\epsilon}{1-\epsilon} d = \frac{1}{1-\epsilon} d,
\end{aligned} \tag{8.4}$$

so that again *(a)* and *(b)* hold. Finally, taking the limit in *(b)* we obtain *(c)*. Therefore, relations *(a)*, *(b)* and *(c)* hold for any infinite subsequence $\mathcal{K}_j \subseteq \left\{\hat{k}, \hat{k}+1, \ldots\right\}$ of index k. □

The hypotheses (H1), (H2) at step $\hat{k}$ are theoretically effective in order to ensure that PSO iteration (2.3) is well suited for problem (1.1). However, (H1) is necessary from Proposition 4.2 and Lemma 4.1, but (H2) may be restrictive for several practical implementations, since it imposes for any $k \geq \hat{k}$ small stepsizes along the direction $\left(p_h^k - x_j^k\right)$ in (2.3). However, *since $\hat{k}$ is arbitrary, the latter drawback is definitely not so relevant.*

Corollary 8.2 *Consider iteration* (2.3) *with position* (2.2) *and suppose Assumption 8.1 holds. Then, for $j = 1, \ldots, P$:*

1. *the sequence $\left\{x_j^k\right\}$ admits limit points and any limit point of $\left\{x_j^k\right\}$ belongs to $\hat{\mathcal{L}}$;*
2. *the sequences $\left\{p_g^k\right\}$ and $\left\{p_j^k\right\}$ admit limit points and any limit point of $\left\{p_g^k\right\}$ and $\left\{p_j^k\right\}$ belongs to $\hat{\mathcal{L}}$;*
3. *if $\left\{x_j^k\right\}_{\mathcal{K}_j}$ is an infinite subsequence of $\left\{x_j^k\right\}$ and $\mathcal{K}_j \subseteq \left\{\hat{k}, \hat{k}+1, \ldots\right\}$, then*

$$\lim_{k\to\infty,\ k\in\mathcal{K}_j} \left\| x_j^{k+1} - x_j^k \right\| = 0. \tag{8.5}$$

Proof From *(a)* of Theorem 8.1 and the compactness of $\hat{\mathcal{L}}$ we obtain *1*. As regards *2*., it is an immediate consequence of *1*. and relations $\left\{p_g^k\right\} \subseteq \left\{p_j^k\right\} \subseteq \left\{x_j^k\right\} \subset \hat{\mathcal{L}}$, for $k \geq \hat{k}$. Finally, (2.3) and *(c)* of Theorem 8.1 give *3*., for $j = 1, \ldots, P$.

Observe that since relation (8.5) holds for *any* infinite subsequence $\mathcal{K}_j$ of $\left\{\hat{k}, \hat{k}+1, \ldots\right\}$, then also the following stronger relation holds:

$$\lim_{k\to\infty} \left\| x_j^{k+1} - x_j^k \right\| = 0, \qquad j = 1, \ldots, P.$$

This result substantially summarizes the convergence properties of the sequences $\left\{x_j^k\right\}$, $j = 1, \ldots, P$, in the compact set $\hat{\mathcal{L}}$. In other words, as long as the hypotheses of Theorem 8.1 hold, we can ensure that when solving (1.1) any subsequence $\left\{x_j^k\right\}_{\mathcal{K}_j} \subseteq \left\{x_j^k\right\}$ remains in $\hat{\mathcal{L}}$ and converges to a point of $\hat{\mathcal{L}}$. We remark that though the latter conclusion requires some conservative conditions on the sequences $\left\{\chi_j^k\right\}$, $\left\{w_j^k\right\}$, $\left\{c_{h,j}^k\right\}$, $\left\{r_{h,j}^k\right\}$, then the convergence of the subsequences of iterates $\left\{x_j^k\right\}$ is a relevant issue in optimization.

9 Conclusions and future work

In this paper a generalized version of Particle Swarm Optimization (PSO) algorithm is analyzed, and some conditions on its coefficients arise from our study. Using an open-loop model for a dynamic linear reformulation of PSO iteration, we provide general conditions in order to possibly avoid diverging paths of the particles. Moreover, based on the numerical experience we provide, our choice for the initial population of PSO, may yield a satisfactory approximation of a global minimum for problem (1.1), even when just a few function evaluations are allowed. Furthermore, when no assumption holds for the function $f(x)$, our preliminary convergence analysis investigates asymptotic properties of the particles trajectories.

An extension to constrained optimization problems with special structure of the feasible set seems the further necessary step. Indeed, in our opinion the PSO literature often contains examples of constrained problems where the use of heuristics covers up both aspects and limits of PSO iteration. We are also concerned with considering further generalizations of iteration (2.3) by including an investigation in the context of *derivative free* methods, not to mention the more general context of *direct search* methods [16] (see also [10]).

Finally, the study of global convergence for PSO iteration and the possibility of clustering the particles, in order to detect multiple global minima, may be other issues of interest.

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Appendix

See Tables 2, 3, 4

Table 2 Numerical results where at most $20n + 1$ function evaluations are allowed, where n is the number of unknowns

Function		x_{rand}	x_{orth}
1	*f**	−1.0316	
	nf_av	41.0000	41.0000
	f_bst	−1.0311	−1.0311
	f_av	−0.7620	**−1.0046**
	f_wst	2.2623	**−0.9884**
	st. dev.	0.3906	0.0115
2	*f**	0.0000	
	nf_av	41.0000	40.9673
	f_bst	0.0000	0.0000
	f_av	0.0080	**0.0019**
	f_wst	1.1663	**0.0707**
	st. dev.	0.0150	0.0076
3	*f**	−0.3523	
	nf_av	41.0000	41.0000
	f_bst	−0.3501	**−0.3520**
	f_av	9.0978	**−0.2348**
	f_wst	265.1136	**−0.0173**
	st. dev.	36.7651	0.0415
4	*f**	−186.7309	
	nf_av	41.0000	41.0000
	f_bst	**−186.3555**	−166.1717
	f_av	−39.4346	**−114.4941**
	f_wst	−16.4318	**−29.6291**
	st. dev.	14.3191	22.9223
5	*f**	−186.7309	
	nf_av	41.0000	41.0000
	f_bst	**−185.4442**	−178.9778

Table 2 continued

Function		x_{rand}	x_{orth}
	f_av	−90.5942	**−107.4875**
	f_wst	1.7588	**−8.9625**
	st. dev.	41.6913	38.6222
6	*f**	−186.7309	
	nf_av	41.0000	41.0000
	f_bst	−126.5949	**−184.7899**
	f_av	1.6031	**−68.5010**
	f_wst	66.6152	**1.1413**
	st. dev.	31.9956	49.8658
7	*f**	−10.1532	
	nf_av	81.0000	81.0000
	f_bst	**−5.4671**	−4.5439
	f_av	−1.1109	**−3.0926**
	f_wst	−0.4649	**−1.0249**
	st. dev.	0.2503	0.3101
8	*f**	−10.4029	
	nf_av	81.0000	81.0000
	f_bst	**−9.5617**	−6.1270
	f_av	**−3.2987**	−2.3605
	f_wst	**−0.9717**	−0.8195
	st. dev.	0.4335	0.7734
9	*f**	−10.5364	
	nf_av	81.0000	81.0000
	f_bst	**−7.0024**	−5.0023
	f_av	**−2.1692**	−1.4074
	f_wst	**−0.9681**	−0.6035
	st. dev.	0.1790	0.0814
10	*f**	−1.0000	
	nf_av	41.0000	41.0000
	f_bst	−0.9987	**−1.0000**
	f_av	−0.4793	**−0.9284**
	f_wst	-0.0001	**−0.7301**
	st. dev.	0.3805	0.0944
11	*f**	−1.0000	
	nf_av	81.0000	81.0000
	f_bst	−0.9584	**−0.9966**
	f_av	−0.0865	**−0.9037**
	f_wst	−0.0005	**−0.7414**
	st. dev.	0.0450	0.0266
12	*f**	−0.2000	
	nf_av	41.0000	41.0000
	f_bst	−0.1995	**−0.2000**
	f_av	0.3024	**−0.1430**
	f_wst	0.8633	**−0.0518**

Table 2 continued

Function		x_{rand}	x_{orth}
	st. dev.	0.3157	0.0674
13	f^*	−0.4000	
	nf_av	81.0000	81.0000
	f_bst	−0.3703	**−0.3987**
	f_av	−0.1968	**−0.3796**
	f_wst	0.2437	**−0.2261**
	st. dev.	0.0105	0.0030
14	f^*	−3.8627	
	nf_av	61.0000	61.0000
	f_bst	−3.8583	**−3.8598**
	f_av	**−2.1876**	−1.1771
	f_wst	**−0.9970**	−0.8269
	st. dev.	0.7150	0.5137
15	f^*	−3.3223	
	nf_av	121.0000	121.0000
	f_bst	−3.2603	**–3.3071**
	f_av	**–2.8489**	−2.2225
	f_wst	**−1.9582**	−1.3843
	st. dev.	0.1136	0.1262
16	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	0.0083	**0.0006**
	f_av	12.6255	**0.4986**
	f_wst	15.2591	**3.1998**
	st. dev.	3.7528	0.3995
17	f^*	0.0000	
	nf_av	101.0000	101.0000
	f_bst	0.0614	**0.0105**
	f_av	2.5774	**0.0606**
	f_wst	4.7692	**0.6100**
	st. dev.	0.4255	0.0123
18	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.5048	**0.0072**
	f_av	3.1842	**0.0223**
	f_wst	6.6074	**0.1331**
	st. dev.	0.0785	0.0000
19	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	3.0340	**0.0151**
	f_av	9.2376	**0.0447**
	f_wst	12.8641	**0.0673**
	st. dev.	0.0022	0.0000
20	f^*	0.0000	

Table 2 continued

Function		x_{rand}	x_{orth}
	nf_av	41.0000	41.0000
	f_bst	0.0243	**0.0050**
	f_av	73.4492	**1.1636**
	f_wst	176.9833	**6.2961**
	st. dev.	39.3344	1.3323
21	f^*	0.0000	
	nf_av	101.0000	101.0000
	f_bst	2.2152	**0.3314**
	f_av	12.9919	**5.4009**
	f_wst	117.5434	**5.5341**
	st. dev.	1.4637	0.6551
22	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	10.0355	**0.1794**
	f_av	40.1507	**2.1021**
	f_wst	83.7101	**3.8478**
	st. dev.	0.0457	0.0434
23	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	42.8864	**1.7102**
	f_av	147.6341	**3.5229**
	f_wst	203.4730	**4.4420**
	st. dev.	0.0134	0.0006
24	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	0.0005	**0.0002**
	f_av	0.1111	**0.0462**
	f_wst	2.5250	**0.1201**
	st. dev.	0.1581	0.0206
25	f^*	0.0000	
	nf_av	101.0000	101.0000
	f_bst	0.0876	**0.0247**
	f_av	0.6975	**0.1079**
	f_wst	7.9700	**0.4511**
	st. dev.	0.0891	0.0202
26	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.5880	**0.0811**
	f_av	4.8455	**0.5370**
	f_wst	8.0153	**0.8211**
	st. dev.	0.0670	0.0019
27	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	4.9139	**0.6706**

Table 2 continued

Function		x_{rand}	x_{orth}
	f_av	13.3446	**0.8036**
	f_wst	32.5649	**3.1085**
	st. dev.	0.0007	0.0008
28	f^*	0.0000	
	nf_av	41.0000	40.9200
	f_bst	0.0114	**0.0000**
	f_av	0.3418	**0.1437**
	f_wst	1.0209	**0.2548**
	st. dev.	0.1416	0.1227
29	f^*	0.0000	
	nf_av	101.0000	101.0000
	f_bst	0.0251	**0.0018**
	f_av	0.4562	**0.0091**
	f_wst	0.9783	**0.1429**
	st. dev.	0.0681	0.0004
30	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.5562	**0.0057**
	f_av	0.8485	**0.0332**
	f_wst	1.1650	**0.0731**
	st. dev.	0.0067	0.0004
31	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	1.2360	**0.0016**
	f_av	1.3872	**0.0022**
	f_wst	1.7438	**0.0653**
	st. dev.	0.0001	0.0000
32	f^*	0.0000	
	nf_av	601.0000	601.0000
	f_bst	3.0273	**0.0268**
	f_av	8.9546	**0.0483**
	f_wst	13.6678	**0.0942**
	st. dev.	0.0000	0.0000
33	f^*	0.0000	
	nf_av	601.0000	601.0000
	f_bst	53.8192	**1.1428**
	f_av	107.3033	**3.4678**
	f_wst	299.5744	**3.9709**
	st. dev.	0.0001	0.0000
34	f^*	0.0000	
	nf_av	601.0000	601.0000
	f_bst	14.4646	**3.1471**
	f_av	31.7934	**3.3890**
	f_wst	60.5632	**3.5046**

Table 2 continued

Function		x_{rand}	x_{orth}
	st. dev.	0.0002	0.0000
35	f^*	0.0000	
	nf_av	601.0000	601.0000
	f_bst	1.5631	**0.0007**
	f_av	2.1459	**0.0389**
	f_wst	2.8092	**0.0710**
	st. dev.	0.0000	0.0000
36	f^*	0.0000	
	nf_av	38.0943	2.0000
	f_bst	0.0000	0.0000
	f_av	0.0067	**0.0000**
	f_wst	0.0743	**0.0000**
	st. dev.	0.0208	0.0000
37	f^*	−24.0000	
	nf_av	41.0000	41.0000
	f_bst	**−175.5378**	-13.8888
	f_av	−11.4022	**−13.7799**
	f_wst	−4.1970	**−13.6092**
	st. dev.	4.1105	0.0384
38	f^*	0.0000	
	nf_av	37.1756	2.0000
	f_bst	0.0000	0.0000
	f_av	0.0536	**0.0000**
	f_wst	0.0715	**0.0000**
	st. dev.	0.0301	0.0000
39	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	0.0052	**0.0003**
	f_av	4.9831	**0.4147**
	f_wst	16.2570	**4.0573**
	st. dev.	6.4970	0.6778
40	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	**3.0046**	5.3873
	f_av	243.0864	**122.1599**
	f_wst	10159.9941	**181.7565**
	st. dev.	547.7312	70.2266
41	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	**2.3153**	48.9850
	f_av	147.4124	**61.4673**
	f_wst	39081.2193	**99.9303**
	st. dev.	672.5250	9.5323
42	f^*	0.0000	

Table 2 continued

Function		x_{rand}	x_{orth}
	nf_av	13.4400	41.0000
	f_bst	**−1.5095**	−0.3244
	f_av	**−0.4419**	0.1633
	f_wst	**−0.0241**	0.7615
	st. dev.	0.4490	0.1330
43	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	0.0096	**0.0000**
	f_av	0.1232	**0.0051**
	f_wst	0.6309	**0.2203**
	st. dev.	0.1447	0.0181
44	f^*	0.0000	
	nf_av	101.0000	101.0000
	f_bst	0.1857	**0.0012**
	f_av	0.5708	**0.0476**
	f_wst	0.9813	**0.3820**
	st. dev.	0.0245	0.0092
45	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.5765	**0.0184**
	f_av	0.6629	**0.0572**
	f_wst	1.2421	**0.1152**
	st. dev.	0.0021	0.0000
46	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	1.2006	**0.0313**
	f_av	1.2514	**0.0838**
	f_wst	1.8996	**0.2689**
	st. dev.	0.0001	0.0000
47	f^*	0.0000	
	nf_av	41.0000	41.0000
	f_bst	**0.0000**	0.0001
	f_av	0.4808	**0.0045**
	f_wst	7.2159	**0.3641**
	st. dev.	0.4173	0.0212
48	f^*	−0.2000	
	nf_av	41.0000	41.0000
	f_bst	**−0.1961**	−0.1930
	f_av	0.3366	**−0.1107**
	f_wst	2.9275	**0.5378**
	st. dev.	0.5145	0.1533
49	f^*	−0.4000	
	nf_av	81.0000	81.0000
	f_bst	−0.3723	**−0.3904**

Table 2 continued

Function		x_{rand}	x_{orth}
	f_av	−0.0146	**−0.3229**
	f_wst	2.0630	**0.5030**
	st. dev.	0.0619	0.0862
50	f^*	−1.0000	
	nf_av	41.0000	41.0000
	f_bst	−0.9997	−0.9997
	f_av	−0.5917	**−0.9660**
	f_wst	0.0000	**−0.7346**
	st. dev.	0.3717	0.0807
51	f^*	−1.0000	
	nf_av	81.0000	81.0000
	f_bst	−0.8912	**−0.9927**
	f_av	−0.8487	**−0.9480**
	f_wst	0.0000	**−0.7884**
	st. dev.	0.1387	0.0039
52	f^*	−450.0000	
	nf_av	201.0000	201.0000
	f_bst	523.0990	**18.3857**
	f_av	**1489.6126**	2224.0132
	f_wst	10823.0498	**8772.2768**
	st. dev.	24.0354	162.8359
53	f^*	−450.0000	
	nf_av	201.0000	201.0000
	f_bst	2757.2050	**1795.0881**
	f_av	14348.2473	**12534.8757**
	f_wst	33420.4572	**14787.2483**
	st. dev.	5.3168	167.1031
54	f^*	−450.0000	
	nf_av	201.0000	201.0000
	f_bst	**********	**********
	f_av	**********	**********
	f_wst	**********	**********
	st. dev.	**********	**********
55	f^*	−450.0000	
	nf_av	201.0000	201.0000
	f_bst	**3089.9603**	3455.1480
	f_av	**6081.8406**	10089.8276
	f_wst	**24748.0692**	30794.4122
	st. dev.	213.3383	49.8110
56	f^*	−310.0000	
	nf_av	201.0000	201.0000
	f_bst	1573.6039	**1027.3114**
	f_av	6939.0372	**5102.0504**
	f_wst	**9761.4497**	14442.8185

Table 2 continued

Function		x_{rand}	x_{orth}
	st. dev.	88.4020	153.8304
57	f^*	390.0000	
	nf_av	201.0000	201.0000
	f_bst	**********	**********
	f_av	**********	**********
	f_wst	**********	**********
	st. dev.	**********	**********
58	f^*	−180.0000	
	nf_av	201.0000	201.0000
	f_bst	-148.0901	**−158.8946**
	f_av	-34.5522	**−45.1808**
	f_wst	608.5574	**252.9588**
	st. dev.	1.9858	1.1555
59	f^*	−140.0000	
	nf_av	201.0000	201.0000
	f_bst	−119.3006	**−119.6066**
	f_av	**−118.9600**	−118.9558
	f_wst	−118.8439	**−118.8716**
	st. dev.	0.0027	0.0003
60	f^*	−330.0000	
	nf_av	201.0000	201.0000
	f_bst	**−288.4127**	−277.0049
	f_av	−244.8765	**−264.7315**
	f_wst	−208.6248	**−234.1968**
	st. dev.	1.0825	0.0133
61	f^*	−330.0000	
	nf_av	201.0000	201.0000
	f_bst	−268.5207	**−271.6312**
	f_av	−259.1338	**−271.6018**
	f_wst	−169.8967	**−175.0173**
	st. dev.	0.5753	**1.1351**

Table 3 Numerical results where at most $40n + 1$ function evaluations are allowed, where n is the number of unknowns

Function		x_{rand}	x_{orth}
1	f^*	−1.0316	
	nf_av	80.9988	80.9997
	f_bst	−1.0316	−1.0316
	f_av	−1.0256	**−1.0315**
	f_wst	6.7013	**−1.0144**
	st. dev.	0.0247	0.0001
2	f^*	0.0000	
	nf_av	81.0000	76.1534

Table 3 continued

Function		x_{rand}	x_{orth}
	f_bst	0.0000	0.0000
	f_av	0.0006	**0.0000**
	f_wst	1.2301	**0.0008**
	st. dev.	0.0066	0.0000
3	*f**	−0.3523	
	nf_av	81.0000	80.9997
	f_bst	−0.3524	−0.3524
	f_av	−0.0504	**−0.1767**
	f_wst	19.8419	**0.1261**
	st. dev.	1.6827	0.0554
4	*f**	−186.7309	
	nf_av	81.0000	81.0000
	f_bst	**−186.7177**	−186.0454
	f_av	**−185.6213**	−57.7841
	f_wst	−10.5231	**−16.8074**
	st. dev.	4.7083	50.2379
5	*f**	−186.7309	
	nf_av	81.0000	81.0000
	f_bst	**−185.8681**	−174.1335
	f_av	**−131.8944**	−115.2027
	f_wst	42.1920	**−24.4475**
	st. dev.	16.4927	15.4110
6	*f**	−186.7309	
	nf_av	81.0000	81.0000
	f_bst	**−185.6830**	−185.5096
	f_av	−38.3881	**−117.3538**
	f_wst	**−23.2163**	2.5344
	st. dev.	25.2581	11.6202
7	*f**	−10.1532	
	nf_av	161.0000	161.0000
	f_bst	**−9.9328**	−5.0501
	f_av	**−9.0034**	−4.5303
	f_wst	−0.8810	**−2.6757**
	st. dev.	0.4926	0.0532
8	*f**	−10.4029	
	nf_av	161.0000	161.0000
	f_bst	**−9.5972**	−9.5345
	f_av	−2.7206	**−4.6912**
	f_wst	**−1.2135**	−1.0022
	st. dev.	0.0361	0.2385
9	*f**	−10.5364	
	nf_av	161.0000	161.0000

Table 3 continued

Function		x_{rand}	x_{orth}
	f_bst	**−10.5021**	−10.0968
	f_av	−1.8374	**−4.7101**
	f_wst	−1.8227	**−1.9925**
	st. dev.	0.1665	0.0317
10	f^*	−1.0000	
	nf_av	81.0000	80.9597
	f_bst	−1.0000	−1.0000
	f_av	**−0.9957**	−0.9932
	f_wst	−0.1482	**−0.9826**
	st. dev.	0.0498	0.0046
11	f^*	−1.0000	
	nf_av	161.0000	161.0000
	f_bst	−0.9959	**−0.9996**
	f_av	−0.9833	**−0.9854**
	f_wst	−0.1036	**−0.9297**
	st. dev.	0.0165	0.0008
12	f^*	−0.2000	
	nf_av	81.0000	79.9200
	f_bst	−0.2000	−0.2000
	f_av	0.3657	**−0.1999**
	f_wst	0.4100	**−0.0522**
	st. dev.	0.0845	0.0004
13	f^*	−0.4000	
	nf_av	161.0000	161.0000
	f_bst	−0.3932	**−0.4000**
	f_av	−0.3788	**−0.3995**
	f_wst	−0.0643	**−0.3909**
	st. dev.	0.0108	0.0000
14	f^*	−3.8627	
	nf_av	121.0000	121.0000
	f_bst	**−3.8625**	−3.8525
	f_av	**−3.8448**	−1.0507
	f_wst	**−0.9990**	−0.9445
	st. dev.	0.0265	0.5310
15	f^*	−3.3223	
	nf_av	241.0000	241.0000
	f_bst	**−3.3202**	−3.3172
	f_av	**−3.2859**	−3.2737
	f_wst	**−3.0404**	−2.9224
	st. dev.	0.0000	0.0058
16	f^*	0.0000	
	nf_av	81.0000	81.0000

Table 3 continued

Function		x_{rand}	x_{orth}
	f_bst	0.0000	0.0000
	f_av	0.0518	**0.0159**
	f_wst	20.6331	**1.5957**
	st. dev.	0.6732	0.0357
17	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.0090	**0.0004**
	f_av	1.5590	**0.0007**
	f_wst	3.7611	**0.1011**
	st. dev.	0.0502	0.0000
18	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.1124	**0.0013**
	f_av	1.9712	**0.0018**
	f_wst	5.2726	**0.0153**
	st. dev.	0.0007	0.0000
19	f^*	0.0000	
	nf_av	801.0000	801.0000
	f_bst	1.5901	**0.0027**
	f_av	2.1843	**0.0192**
	f_wst	6.7800	**0.0288**
	st. dev.	0.0000	0.0000
20	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	0.0062	**0.0041**
	f_av	**1.8047**	2.8294
	f_wst	145.5811	**9.7906**
	st. dev.	4.9618	0.7430
21	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.1001	**0.0036**
	f_av	2.5519	**0.0051**
	f_wst	27.7203	**1.8894**
	st. dev.	0.0917	0.0372
22	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	2.4251	**0.0267**
	f_av	20.7709	**0.4657**
	f_wst	26.5295	**2.3463**
	st. dev.	0.0112	0.0009
23	f^*	0.0000	
	nf_av	801.0000	801.0000

Table 3 continued

Function		x_{rand}	x_{orth}
	f_bst	23.8019	**0.1512**
	f_av	52.0887	**3.3690**
	f_wst	104.3099	**4.4062**
	st. dev.	0.0000	0.0000
24	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	**0.0000**	0.0002
	f_av	**0.0154**	0.0787
	f_wst	2.5431	**0.1246**
	st. dev.	0.0851	0.0371
25	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.0061	**0.0012**
	f_av	0.0530	**0.0029**
	f_wst	2.0578	**0.2862**
	st. dev.	0.0113	0.0001
26	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.2874	**0.0057**
	f_av	2.0616	**0.1826**
	f_wst	5.2402	**0.3173**
	st. dev.	0.0011	0.0001
27	f^*	0.0000	
	nf_av	801.0000	801.0000
	f_bst	3.6909	**0.1282**
	f_av	3.6909	**2.5818**
	f_wst	34.6621	**2.8836**
	st. dev.	0.0000	0.0000
28	f^*	0.0000	
	nf_av	81.0000	80.9995
	f_bst	0.0000	0.0000
	f_av	0.1352	**0.1258**
	f_wst	0.9610	**0.1956**
	st. dev.	0.0873	0.0606
29	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.0103	**0.0003**
	f_av	0.2176	**0.0016**
	f_wst	0.4950	**0.0857**
	st. dev.	0.0054	0.0031
30	f^*	0.0000	
	nf_av	401.0000	401.0000

Table 3 continued

Function		x_{rand}	x_{orth}
	f_bst	0.1674	**0.0010**
	f_av	0.2930	**0.0089**
	f_wst	0.8117	**0.0150**
	st. dev.	0.0001	0.0000
31	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.6716	**0.0006**
	f_av	1.2212	**0.0072**
	f_wst	1.4395	**0.0157**
	st. dev.	0.0000	0.0000
32	*f**	0.0000	
	nf_av	1201.0000	1201.0000
	f_bst	2.3918	**0.0086**
	f_av	3.2339	**0.0188**
	f_wst	11.8370	**0.0283**
	st. dev.	0.0000	0.0000
33	*f**	0.0000	
	nf_av	1201.0000	1201.0000
	f_bst	46.6585	**0.3253**
	f_av	90.1271	**3.2422**
	f_wst	212.8171	**3.7684**
	st. dev.	0.0000	0.0000
34	*f**	0.0000	
	nf_av	1201.0000	1201.0000
	f_bst	14.3632	**3.0534**
	f_av	24.0399	**3.2849**
	f_wst	43.5382	**3.3398**
	st. dev.	0.0000	0.0000
35	*f**	0.0000	
	nf_av	1201.0000	1201.0000
	f_bst	1.1964	**0.0007**
	f_av	1.6255	**0.0036**
	f_wst	1.7549	**0.0383**
	st. dev.	0.0000	0.0000
36	*f**	0.0000	
	nf_av	57.4598	2.0000
	f_bst	0.0000	0.0000
	f_av	0.0000	0.0000
	f_wst	0.0000	0.0000
	st. dev.	0.0000	0.0000
37	*f**	−24.0000	
	nf_av	81.0000	81.0000
	f_bst	−13.8887	**−13.8888**

Table 3 continued

Function		x_{rand}	x_{orth}
	f_av	−13.8343	**−13.8888**
	f_wst	−11.8549	**−13.8848**
	st. dev.	0.0166	0.0000
38	f^*	0.0000	
	nf_av	73.4970	2.0000
	f_bst	0.0000	0.0000
	f_av	0.0005	**0.0000**
	f_wst	0.0714	**0.0000**
	st. dev.	0.0047	0.0000
39	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	0.0000	0.0000
	f_av	**0.2077**	0.2284
	f_wst	12.3695	**3.3852**
	st. dev.	0.3362	0.2612
40	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	**3.0001**	3.1575
	f_av	3397.8781	**91.0935**
	f_wst	8893.6913	**92.0352**
	st. dev.	1633.4400	6.2081
41	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	**0.0060**	3.8635
	f_av	179.5016	**51.0745**
	f_wst	418.4434	**78.6621**
	st. dev.	24.0158	1.0640
42	f^*	0.0000	
	nf_av	75.9793	79.5200
	f_bst	**−3.9341**	−0.1532
	f_av	**−0.0104**	0.0070
	f_wst	1.2303	**0.6448**
	st. dev.	0.3967	0.0362
43	f^*	0.0000	
	nf_av	81.0000	81.0000
	f_bst	0.0000	0.0000
	f_av	0.2317	**0.1642**
	f_wst	0.6313	**0.1980**
	st. dev.	0.1715	0.0711
44	f^*	0.0000	
	nf_av	201.0000	201.0000
	f_bst	0.0823	**0.0002**

Table 3 continued

Function		x_{rand}	x_{orth}
	f_av	0.2185	**0.0002**
	f_wst	0.6945	**0.3543**
	st. dev.	0.0157	0.0003
45	f^*	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.1216	**0.0017**
	f_av	0.5418	**0.0046**
	f_wst	0.9413	**0.0585**
	st. dev.	0.0001	0.0000
46	f^*	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.5762	**0.0125**
	f_av	0.9408	**0.0372**
	f_wst	1.2876	**0.1021**
	st. dev.	0.0000	0.0000
47	f^*	0.0000	
	nf_av	80.9999	81.0000
	f_bst	0.0000	0.0000
	f_av	**0.0038**	0.0544
	f_wst	4.1013	**0.5303**
	st. dev.	0.0231	0.1221
48	f^*	−0.2000	
	nf_av	81.0000	81.0000
	f_bst	−0.2000	−0.2000
	f_av	0.2780	**0.0422**
	f_wst	2.7894	**0.5352**
	st. dev.	0.2189	0.2110
49	f^*	−0.4000	
	nf_av	161.0000	161.0000
	f_bst	−0.3981	**−0.3988**
	f_av	**−0.3409**	−0.2477
	f_wst	0.9834	**0.3712**
	st. dev.	0.0084	0.0123
50	f^*	−1.0000	
	nf_av	81.0000	81.0000
	f_bst	−0.9999	**−1.0000**
	f_av	−0.1864	**−0.9984**
	f_wst	0.0000	**−0.9373**
	st. dev.	0.3648	0.0033
51	f^*	−1.0000	
	nf_av	161.0000	161.0000
	f_bst	−0.9956	**−1.0000**
	f_av	−0.9639	**−0.9998**

Table 3 continued

Function		x_{rand}	x_{orth}
	f_wst	−0.0883	**−0.8835**
	st. dev.	0.0249	0.0021
52	f^*	−450.0000	
	nf_av	401.0000	401.0000
	f_bst	−325.7800	**−369.7007**
	f_av	**−199.9334**	233.0814
	f_wst	**1895.5326**	1986.8290
	st. dev.	0.7171	1.0119
53	f^*	−450.0000	
	nf_av	401.0000	401.0000
	f_bst	1548.3998	**−142.8780**
	f_av	**3991.6247**	6170.0695
	f_wst	15897.7291	**10069.2206**
	st. dev.	0.8376	0.3024
54	f^*	−450.0000	
	nf_av	401.0000	401.0000
	f_bst	**********	**********
	f_av	**********	**********
	f_wst	**********	**********
	st. dev.	10425.1447	32799.9693
55	f^*	−450.0000	
	nf_av	401.0000	401.0000
	f_bst	**730.6799**	810.4169
	f_av	**6879.8607**	8522.0477
	f_wst	20073.7803	**15377.7628**
	st. dev.	1.3513	4.3252
56	f^*	−310.0000	
	nf_av	401.0000	401.0000
	f_bst	**1038.4336**	1056.9945
	f_av	5588.8400	**2481.8093**
	f_wst	**5588.8401**	5840.6768
	st. dev.	0.1341	0.4032
57	f^*	390.0000	
	nf_av	401.0000	401.0000
	f_bst	99141.6079	**9506.7645**
	f_av	**********	**********
	f_wst	**********	**********
	st. dev.	3462.8607	59.7758
58	f^*	−180.0000	
	nf_av	401.0000	401.0000
	f_bst	−176.1459	**−177.0556**

Table 3 continued

Function		x_{rand}	x_{orth}
	f_av	−155.9399	**−161.8662**
	f_wst	77.7073	**−115.5290**
	st. dev.	0.0737	0.0025
59	*f**	−140.0000	
	nf_av	401.0000	401.0000
	f_bst	−119.3700	**−119.6343**
	f_av	**−119.1271**	−119.0583
	f_wst	**−118.9100**	−118.8962
	st. dev.	0.0001	0.0001
60	*f**	−330.0000	
	nf_av	401.0000	401.0000
	f_bst	**−303.9368**	−293.0955
	f_av	**−284.6053**	−259.1401
	f_wst	−221.5411	**−257.9441**
	st. dev.	0.0089	0.0145
61	*f**	−330.0000	
	nf_av	401.0000	401.0000
	f_bst	**−290.4764**	−268.4363
	f_av	**−254.5831**	−228.5617
	f_wst	**−228.4065**	−185.2952
	st. dev.	0.0009	0.0016

Table 4 Numerical results where at most $80n + 1$ function evaluations are allowed, where n is the number of unknowns

Function		x_{rand}	x_{orth}
1	*f**	−1.0316	
	nf_av	160.9999	154.2706
	f_bst	−1.0316	−1.0316
	f_av	−1.0003	**−1.0289**
	f_wst	−0.0869	**−0.9429**
	st. dev.	0.1532	0.0120
2	*f**	0.0000	
	nf_av	160.9999	160.9908
	f_bst	0.0000	0.0000
	f_av	0.1491	**0.0107**
	f_wst	0.6786	**0.0111**
	st. dev.	0.0291	0.0021
3	*f**	−0.3523	
	nf_av	153.9779	157.6392
	f_bst	−0.3524	−0.3524
	f_av	0.3143	**−0.3463**
	f_wst	15.4158	**−0.1524**
	st. dev.	2.2360	0.0332

Table 4 continued

Function		x_{rand}	x_{orth}
4	*f**	−186.7309	
	nf_av	161.0000	161.0000
	f_bst	**−186.7309**	−186.7308
	f_av	**−184.2080**	−120.4618
	f_wst	**−40.8822**	−24.2326
	st. dev.	12.6845	15.9691
5	*f**	−186.7309	
	nf_av	161.0000	161.0000
	f_bst	−183.7005	**−186.7308**
	f_av	−36.3514	**−185.6485**
	f_wst	−12.8432	**−16.7213**
	st. dev.	25.2531	6.8797
6	*f**	−186.7309	
	nf_av	161.0000	161.0000
	f_bst	−186.2832	**−186.7289**
	f_av	−130.6094	**−184.1674**
	f_wst	25.9599	**−1.4774**
	st. dev.	10.6479	12.9823
7	*f**	−10.1532	
	nf_av	321.0000	321.0000
	f_bst	**−10.1486**	−9.9500
	f_av	−2.6828	**−4.9913**
	f_wst	−2.5415	**−4.5535**
	st. dev.	0.0000	0.0362
8	*f**	−10.4029	
	nf_av	321.0000	321.0000
	f_bst	**−10.4023**	−10.3997
	f_av	−2.7512	**−10.3994**
	f_wst	−2.7380	**−5.0118**
	st. dev.	0.0000	0.0388
9	*f**	−10.5364	
	nf_av	321.0000	321.0000
	f_bst	**−10.5362**	−10.4462
	f_av	−2.8062	**−5.1257**
	f_wst	−1.6747	**−3.8351**
	st. dev.	0.0075	0.0001
10	*f**	−1.0000	
	nf_av	160.9109	160.9021
	f_bst	−1.0000	−1.0000
	f_av	−0.9566	**−1.0000**
	f_wst	0.0000	**−0.9527**
	st. dev.	0.0309	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
11	*f**	−1.0000	
	nf_av	321.0000	320.9999
	f_bst	−1.0000	−1.0000
	f_av	−0.9993	**−0.9996**
	f_wst	−0.0007	**−0.9970**
	st. dev.	0.0000	0.0000
12	*f**	−0.2000	
	nf_av	160.9999	160.9995
	f_bst	−0.2000	−0.2000
	f_av	0.0847	**−0.1945**
	f_wst	0.3874	**−0.0522**
	st. dev.	0.0546	0.0273
13	*f**	−0.4000	
	nf_av	321.0000	321.0000
	f_bst	−0.4000	−0.4000
	f_av	−0.3999	**−0.4000**
	f_wst	−0.2098	**−0.3968**
	st. dev.	0.0005	0.0000
14	*f**	−3.8627	
	nf_av	240.9899	241.0000
	f_bst	−3.8627	−3.8627
	f_av	**−3.0909**	−1.0008
	f_wst	**−1.0008**	−0.9209
	st. dev.	0.0289	0.0000
15	*f**	−3.3223	
	nf_av	481.0000	481.0000
	f_bst	−3.3223	−3.3223
	f_av	−2.6686	**−3.2021**
	f_wst	−2.6434	**−3.1635**
	st. dev.	0.0001	0.0000
16	*f**	0.0000	
	nf_av	160.9555	161.0000
	f_bst	0.0000	0.0000
	f_av	14.6502	**0.0011**
	f_wst	20.7039	**1.5554**
	st. dev.	2.5605	0.0016
17	*f**	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.0000	0.0000
	f_av	0.0070	**0.0001**
	f_wst	1.3369	**0.0010**
	st. dev.	0.0000	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
18	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.0016	**0.0000**
	f_av	0.1618	**0.0000**
	f_wst	1.3392	**0.0029**
	st. dev.	0.0000	0.0000
19	*f**	0.0000	
	nf_av	1601.0000	1601.0000
	f_bst	0.1350	**0.0003**
	f_av	0.3756	**0.0015**
	f_wst	3.9059	**0.0042**
	st. dev.	0.0000	0.0000
20	*f**	0.0000	
	nf_av	161.0000	161.0000
	f_bst	0.0000	0.0000
	f_av	38.9960	**0.0594**
	f_wst	133.1258	**7.7738**
	st. dev.	7.0459	0.2920
21	*f**	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.0005	**0.0000**
	f_av	0.0033	**0.0000**
	f_wst	30.1312	**5.6069**
	st. dev.	0.0001	0.0000
22	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.1346	**0.0004**
	f_av	6.4987	**0.0056**
	f_wst	57.6703	**2.7999**
	st. dev.	0.0000	0.0000
23	*f**	0.0000	
	nf_av	1601.0000	1601.0000
	f_bst	5.1918	**0.0121**
	f_av	17.0401	**3.2600**
	f_wst	49.8667	**3.6099**
	st. dev.	0.0000	0.0000
24	*f**	0.0000	
	nf_av	161.0000	161.0000
	f_bst	0.0000	0.0000
	f_av	**0.0012**	0.0032
	f_wst	2.4976	**0.1103**
	st. dev.	0.0041	0.0021

Table 4 continued

Function		x_{rand}	x_{orth}
25	*f**	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.0000	0.0000
	f_av	0.1676	**0.0029**
	f_wst	5.3611	**0.0975**
	st. dev.	0.0004	0.0000
26	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.0679	**0.0002**
	f_av	0.1741	**0.0097**
	f_wst	1.7542	**0.1114**
	st. dev.	0.0000	0.0000
27	*f**	0.0000	
	nf_av	1601.0000	1601.0000
	f_bst	0.8071	**0.1684**
	f_av	12.9959	**1.9259**
	f_wst	13.7846	**2.3050**
	st. dev.	0.0000	0.0000
28	*f**	0.0000	
	nf_av	161.0000	159.9978
	f_bst	0.1456	**0.0000**
	f_av	**0.1457**	0.1861
	f_wst	0.5825	**0.1955**
	st. dev.	0.0017	0.0382
29	*f**	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.0022	**0.0000**
	f_av	0.5592	**0.0000**
	f_wst	0.5889	**0.0016**
	st. dev.	0.0006	0.0000
30	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.0022	**0.0000**
	f_av	0.1564	**0.0002**
	f_wst	0.6262	**0.0010**
	st. dev.	0.0000	0.0000
31	*f**	0.0000	
	nf_av	1601.0000	1601.0000
	f_bst	0.0813	**0.0001**
	f_av	0.7361	**0.0002**
	f_wst	0.7361	**0.0036**
	st. dev.	0.0000	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
32	*f**	0.0000	
	nf_av	2401.0000	2401.0000
	f_bst	0.8929	**0.0012**
	f_av	1.1398	**0.0025**
	f_wst	4.1369	**0.0077**
	st. dev.	0.0000	0.0000
33	*f**	0.0000	
	nf_av	2401.0000	2401.0000
	f_bst	14.3891	**0.1018**
	f_av	91.5556	**2.9880**
	f_wst	136.0398	**3.3736**
	st. dev.	0.0000	0.0000
34	*f**	0.0000	
	nf_av	2401.0000	2401.0000
	f_bst	4.2874	**2.7202**
	f_av	14.2528	**3.0368**
	f_wst	33.5331	**3.1575**
	st. dev.	0.0000	0.0000
35	*f**	0.0000	
	nf_av	2401.0000	2401.0000
	f_bst	0.2889	**0.0002**
	f_av	0.5329	**0.0013**
	f_wst	0.9394	**0.0060**
	st. dev.	0.0000	0.0000
36	*f**	0.0000	
	nf_av	137.5641	1.9600
	f_bst	0.0000	0.0000
	f_av	0.0059	**0.0000**
	f_wst	0.0072	**0.0000**
	st. dev.	0.0027	0.0000
37	*f**	−24.0000	
	nf_av	161.0000	161.0000
	f_bst	−13.8888	−13.8888
	f_av	−13.7937	**−13.8884**
	f_wst	**−5.9218**	−3.3321
	st. dev.	0.0353	0.0614
38	*f**	0.0000	
	nf_av	160.7009	2.0000
	f_bst	0.0000	0.0000
	f_av	0.0022	**0.0000**
	f_wst	0.0714	**0.0000**
	st. dev.	0.0111	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
39	*f**	0.0000	
	nf_av	161.0000	161.0000
	f_bst	0.0000	0.0000
	f_av	**0.0681**	1.4958
	f_wst	7.7934	**1.5551**
	st. dev.	0.3407	0.2919
40	*f**	0.0000	
	nf_av	161.0000	161.0000
	f_bst	3.0000	3.0000
	f_av	29.9812	**18.0002**
	f_wst	4084.3619	**534.4600**
	st. dev.	0.9502	13.6330
41	*f**	0.0000	
	nf_av	161.0000	161.0000
	f_bst	**0.0033**	48.9843
	f_av	125.8512	**65.9482**
	f_wst	142.8702	**95.3607**
	st. dev.	8.1917	3.2844
42	*f**	0.0000	
	nf_av	51.7200	161.0000
	f_bst	**−1.3152**	−0.8090
	f_av	**−0.4473**	0.0004
	f_wst	1.2603	**1.1626**
	st. dev.	0.4337	0.0004
43	*f**	0.0000	
	nf_av	158.0400	160.9999
	f_bst	0.0000	0.0000
	f_av	0.0109	**0.0000**
	f_wst	1.0890	**0.1955**
	st. dev.	0.0240	0.0010
44	*f**	0.0000	
	nf_av	401.0000	401.0000
	f_bst	0.0002	**0.0000**
	f_av	0.1935	**0.0001**
	f_wst	0.4792	**0.2387**
	st. dev.	0.0003	0.0000
45	*f**	0.0000	
	nf_av	801.0000	801.0000
	f_bst	0.0014	**0.0000**
	f_av	0.0068	**0.0001**
	f_wst	0.4230	**0.0027**
	st. dev.	0.0000	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
46	f^*	0.0000	
	nf_av	1601.0000	1601.0000
	f_bst	0.0608	**0.0005**
	f_av	0.0608	**0.0128**
	f_wst	0.8399	**0.0128**
	st. dev.	0.0000	0.0000
47	f^*	0.0000	
	nf_av	160.9995	161.0000
	f_bst	0.0000	0.0000
	f_av	0.0004	**0.0002**
	f_wst	0.8689	**0.1154**
	st. dev.	0.0009	0.0040
48	f^*	−0.2000	
	nf_av	161.0000	158.3200
	f_bst	−0.2000	−0.2000
	f_av	**−0.1679**	0.3638
	f_wst	2.9311	**0.5431**
	st. dev.	0.1485	0.1135
49	f^*	−0.4000	
	nf_av	321.0000	321.0000
	f_bst	−0.4000	−0.4000
	f_av	**−0.2505**	−0.2412
	f_wst	0.5158	**0.4829**
	st. dev.	0.0011	0.0001
50	f^*	−1.0000	
	nf_av	160.8139	157.9922
	f_bst	−1.0000	−1.0000
	f_av	−0.8884	**−1.0000**
	f_wst	0.0000	**−0.9972**
	st. dev.	0.1585	0.0000
51	f^*	−1.0000	
	nf_av	321.0000	321.0000
	f_bst	−1.0000	−1.0000
	f_av	−0.9631	**−0.9999**
	f_wst	−0.7166	**−0.9977**
	st. dev.	0.0003	0.0000
52	f^*	−450.0000	
	nf_av	801.0000	801.0000
	f_bst	−444.6819	**−449.5138**
	f_av	−403.3612	**−445.1387**
	f_wst	−355.1254	**−374.5348**
	st. dev.	0.0000	0.0000

Table 4 continued

Function		x_{rand}	x_{orth}
53	*f**	−450.0000	
	nf_av	801.0000	801.0000
	f_bst	**−265.7922**	−189.7569
	f_av	**340.0342**	1064.0588
	f_wst	5123.3214	**3397.5009**
	st. dev.	0.0002	0.0005
54	*f**	−450.0000	
	nf_av	801.0000	801.0000
	f_bst	**********	**********
	f_av	**********	**********
	f_wst	**********	**********
	st. dev.	10.9457	8.4218
55	*f**	−450.0000	
	nf_av	801.0000	801.0000
	f_bst	−141.3096	**−155.1592**
	f_av	**−141.3096**	9866.8644
	f_wst	20816.6537	**11918.8605**
	st. dev.	0.0041	0.0011
56	*f**	−310.0000	
	nf_av	801.0000	801.0000
	f_bst	33.2944	**−122.7519**
	f_av	**33.2944**	2415.1404
	f_wst	**3459.2775**	5037.3717
	st. dev.	0.0007	0.0004
57	*f**	390.0000	
	nf_av	801.0000	801.0000
	f_bst	751.4259	**681.7288**
	f_av	31364.8093	**6358.9341**
	f_wst	**********	**********
	st. dev.	0.0114	0.0023
58	*f**	−180.0000	
	nf_av	801.0000	801.0000
	f_bst	−178.4040	**−178.8371**
	f_av	**−174.3604**	−161.9260
	f_wst	−115.6607	**−155.1947**
	st. dev.	0.0000	0.0000
59	*f**	−140.0000	
	nf_av	801.0000	801.0000
	f_bst	−119.5378	**−119.5547**
	f_av	**−119.4353**	−119.1089
	f_wst	**−119.0682**	−119.0218

Table 4 continued

Function		x_{rand}	x_{orth}
	st. dev.	0.0000	0.0000
60	f^*	−330.0000	
	nf_av	801.0000	801.0000
	f_bst	**−315.5572**	−314.6545
	f_av	**−301.5512**	−267.8193
	f_wst	**−270.3378**	−260.7737
	st. dev.	0.0000	0.0000
61	f^*	−330.0000	
	nf_av	801.0000	801.0000
	f_bst	**−310.1780**	−298.7246
	f_av	**−271.7641**	−258.0945
	f_wst	**−220.7753**	−213.8389
	st. dev.	0.0000	0.0000

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